Restricted Symmetric Signed Permutations

Andy Hardt
Department of Mathematics
Carleton College
Northfield, MN 55057 USA

hardta@carleton.edu

Justin M. Troyka
Department of Mathematics
Carleton College
Northfield, MN 55057 USA

troykaj@carleton.edu

27 September 2012

Abstract

The symmetry group $D_4 \oplus \mathbb{Z}_2$ acts on the set of signed permutations by rotations, reflections, and bar operations (flip the sign of each entry). Following Egge's work [8] on unsigned permutations, we enumerate the signed permutations that, given a symmetry subgroup H and a set R of length-2 signed patterns, are invariant under H and avoid R. Mansour and West [14] began this work by enumerating the signed permutations that avoid R, not taking symmetries into account. Dukes and Mansour [5] continued by enumerating signed involutions that avoid R. This paper considers the remaining subgroups of $D_4 \oplus \mathbb{Z}_2$, thus completing the enumeration. The resulting sequences include the Catalan numbers and the central binomial coefficients, and many of them are given recursively. A few of the sets can be counted in two different ways, yielding combinatorial identities.

1 Introduction

A signed permutation is a permutation, written in one-line notation, in which each entry may or may not have a bar over it. Equivalently, each entry of a signed permutation carries either a plus (nonbarred) or a minus (barred). Let B_n denote the set of signed permutations of length n. Given $\pi \in B_n$, we can draw a diagram of π in an $n \times n$ grid: for each i, put a dot in row $|\pi(i)|$ of column i (counting from the bottom and from the left, respectively); fill in the dot if $\pi(i)$ is nonbarred and leave the dot empty if $\pi(i)$ is barred.

The symmetry group of the square, D_4, acts on these diagrams and thus on the elements of B_n. We can also apply a bar operation: given $\pi \in B_n$, keep the diagram's dots in the same places but flip the sign of each dot. We write $\mathfrak{S}$ to denote the group generated by D_4 and the bar operation. We observe that $\mathfrak{S}$ is isomorphic to $D_4 \oplus \mathbb{Z}_2$. Within $\mathfrak{S}$, we use e to denote the identity element, we use R_{90}, R_{180}, and R_{270} to denote rotations by 90, 180, and 270 degrees respectively, we use D to denote a reflection across the "/" diagonal (the D diagonal), and we use D' to denote a reflection across the "\" diagonal (the D' diagonal). We will generally view these operations geometrically, but it is worth noting that R_{180} is the reverse–complement map, while D is the group-theoretic inverse. We also observe that the bar operation commutes with everything in $\mathfrak{S}$, so for any $g \in D_4$ we use $\overline{g}$ to denote g composed with the bar operation.

In this paper we will be concerned with signed permutations that are invariant under certain subgroups of $\mathfrak{S}$. For any subgroup $H \leq \mathfrak{S}$, we write B_n^H to denote the set of signed permutations of n that are invariant under the action of H. There are 35 subgroups of $\mathfrak{S}$, but we need not consider all of them. No signed permutations of length ≥ 2 are invariant under any of the following group elements: the vertical and horizontal reflections, either of these composed with the bar operation, and the bar operation itself. Thus, if a subgroup H contains one or more of these, then B_n^H is empty for all $n \geq 2$. After removing all such subgroups from consideration, 13 remain. In addition, some of these are in conjugate pairs, resulting in 10 conjugacy classes:

$$\begin{array}{cc}
\{e\}, & \left[\quad \{e,\overline{D}\} \text{ and } \{e,\overline{D'}\} \quad\right], \\
\left[\quad \{e,D\} \text{ and } \{e,D'\} \quad\right], & \left[\quad \{e,\overline{R_{180}},D,\overline{D'}\} \text{ and } \{e,\overline{R_{180}},\overline{D},D'\} \quad\right], \\
\{e,R_{180}\}, & \{e,R_{180},\overline{D},\overline{D'}\}, \\
\{e,\overline{R_{180}}\}, & \{e,R_{90},R_{180},R_{270}\}, \\
\{e,R_{180},D,D'\}, & \{e,\overline{R_{90}},R_{180},\overline{R_{270}}\}.
\end{array}$$

Symmetry is one of the two main concepts of our paper; the other is pattern restriction, or pattern avoidance. Given signed permutations $\pi \in B_n$ and $\rho \in B_r$, we say that π *contains* ρ whenever π has a substring that has the same relative order and sign configuration as ρ. We say that π *avoids* ρ whenever π does not contain ρ. Given a set R of signed permutations, we say that π avoids R whenever π avoids every element of R. We write $B_n(R)$ to denote the set of signed permutations of n that avoid R. Given a subgroup $H \leq \mathfrak{S}$ and a set R of signed permutations, we write $B_n^H(R)$ to denote the set of signed permutations of n that are invariant under H and avoid R.

The seminal paper on pattern restriction is the work of Simion and Schmidt [17], who enumerated the unsigned permutations that avoid R for every $R \subseteq S_3$, as well as the unsigned involutions that avoid ρ for each single pattern $\rho \in S_3$. Since then, a large body of work has been done on pattern-restricted permutations; the following references are a small sampling of the literature which is most relevant to the present study. Guibert and Mansour [10] enumerated the unsigned involutions that avoid R for every $R \subseteq S_3$, and they did the same for several sets R of longer patterns. Simion [16] and Mansour and West [14] enumerated the signed permutations that avoid R for every $R \subseteq B_2$. Dukes and Mansour [5] did the same thing for signed involutions; Dukes, Mansour, and Reifegerste [6] extended these results to signed involutions avoiding patterns of lengths 3 and 4; Mansour and Sun [12] enumerated restricted signed permutations of even sign (meaning the product of the signs is positive). Egge [8] enumerated the unsigned permutations that are invariant under H and avoid R, for every $H \leq D_4$ and $R \subseteq S_3$.

Our purpose is to present formulas or recurrence relations that enumerate $B_n^H(R)$, for every $H \leq \mathfrak{S}$ and every $R \subseteq B_2$. This is analogous to the work of Egge [8], who enumerated $S_n^H(R)$ for every $H \leq D_4$ and every $R \subseteq S_3$. $B_n^H(R)$ has already been enumerated for $H = \{e\}$ (see [16], [14]) and $H = \{e, D\}$ (see [5]). Thus, in this paper, we enumerate $B_n^H(R)$ only for the remaining 11 subgroups $H \leq \mathfrak{S}$. The following lemma considerably reduces the task of enumeration.

Lemma 1.1 *Let $H \leq \mathfrak{S}$, let $g \in \mathfrak{S}$, and let R be a set of signed patterns. Let $H' = gHg^{-1}$ be conjugate to H by g. For all n, if $\pi \in B_n^H(R)$, then $g(\pi) \in B_n^{H'}(g(R))$. That is, g is a canonical bijection from $B_n^H(R)$ to $B_n^{H'}(g(R))$.*

Proof. Let $h' \in H'$. Then there exists $h \in H$ such that $h' = ghg^{-1}$. Using the fact that π is invariant under h,

$$h'(g(\pi)) = ghg^{-1}g(\pi) = gh(\pi) = g(\pi),$$

so $g(\pi)$ is invariant under h' for every $h' \in H'$. Thus, $g(\pi)$ is invariant under H'.

Furthermore, it is easy to check (by drawing diagrams) that π avoids R if and only if $g(\pi)$ avoids $g(R)$. ∎

This lemma has three important consequences. First of all, let $H \leq \mathfrak{S}$, let R be a set of patterns, and let $\pi \in B_n^H(R)$. Then Lemma 1.1 lets us conclude that $\pi \in B_n^H(h(R))$ for every $h \in H$. This means that π avoids $H(R)$, the image of R under H. Thus, $B_n^H(R) = B_n^H(H(R))$. Given H and R, in this paper we will only explicitly enumerate $B_n^H(H(R))$—that is, we only examine the maximal equivalent pattern set.

Second, if $g \in \mathfrak{S}$ is in $N(H)$ (the normalizer of H), then by Lemma 1.1 g is a canonical bijection from $B_n^H(R)$ to $B_n^H(g(R))$. Thus, $|B_n^H(R)| = |B_n^H(g(R))|$ for all $g \in N(H)$, so we need only check one of them. If H is conjugate only to itself, then $N(H) = \mathfrak{S}$; if H comes in a conjugate pair, then $N(H) = \{e, R_{180}, D, D', \overline{e}, \overline{R_{180}}, \overline{D}, \overline{D'}\}$.

Third, if $H' = gHg^{-1}$ and $H \neq H'$ (that is, $\{H, H'\}$ is one of the three conjugate pairs), then by Lemma 1.1 g is a canonical bijection from $B_n^H(R)$ to $B_n^{H'}(g(R))$. Thus, for every $R \subseteq B_2$, there exists $R' \subseteq B_2$

[namely $R' = g(R)$] such that $|B_n^H(R)| = |B_n^{H'}(R')|$, so we need only check H. The upshot is that it suffices to look at only one representative from each of the 8 remaining conjugacy classes.

We now outline our paper and provide examples of some results. In Section 2, we enumerate pattern-restricted signed permutations invariant under R_{180}. To count the unrestricted R_{180}-invariant permutations, we establish a bijection to four-colored permutations (Thm. 2.1); to count those that avoid $\bar{2}\bar{1}$, we use a similar bijection to three-colored permutations (Thm. 2.2). We find other formulas that enumerate the latter, including a simple recurrence relation that we prove using generating functions. In Section 3, we enumerate pattern-restricted signed permutations invariant under $\overline{R_{180}}$. In order to count the unrestricted permutations (Thm. 3.1) and the permutations that avoid $\bar{2}1$ (Thm. 3.2), we establish bijections to certain sets of permutations invariant under R_{180}. We also use generating trees and the kernel method to prove that the set avoiding $\{\bar{2}\bar{1}, \bar{2}1, 21\}$ is enumerated by $\binom{2k+1}{k}$ (Thm. 3.9). In Section 4, we enumerate pattern-restricted signed permutations invariant under D and D', which generate the subgroup $H = \{e, R_{180}, D, D'\}$. In Section 5, we do the same for $\overline{D}$-invariant permutations. For example, the set avoiding $\{\bar{2}\bar{1}, \bar{2}1, 21\}$ is enumerated by the Catalan numbers (Thm. 5.6). In Section 6, we enumerate pattern-restricted signed permutations invariant under D and $\overline{D'}$, which generate the subgroup $W = \{e, \overline{R_{180}}, D, \overline{D'}\}$. We use a bijection to prove that the unrestricted W-invariant permutations are enumerated by 2^k times the number of unsigned involutions (Thm. 6.1). In order to prove that the set avoiding $\{\bar{1}\bar{2}, \bar{1}2, 12\}$ is enumerated by $\binom{k}{\lfloor k/2 \rfloor}$, we use a bijection to a particular class of restricted unsigned permutations (Thm. 6.6). In Section 7, we enumerate pattern-restricted signed permutations invariant under $\overline{D}$ and $\overline{D'}$, which generate the subgroup $\overline{H} = \{e, R_{180}, \overline{D}, \overline{D'}\}$. In Section 8, we do the same for $\langle R_{90} \rangle$- or $\langle \overline{R_{90}} \rangle$-invariant permutations, both of which we treat in the same section because of their similarity. The paper until this point deals only with length-2 patterns, but in Section 9 we give two general enumerative results about permutations that avoid certain longer patterns. In Section 10, we pose open questions regarding combinatorial proofs, r-colored permutations, and connections with permutation tableaux.

2 Signed permutations invariant under R_{180}

Let B_n^{180} be the set of signed permutations of length n that are invariant under the subgroup $\langle R_{180} \rangle = \{e, R_{180}\}$ of $\mathfrak{S}$. Recall from our introduction that applying R_{180} to a signed permutation is the same as taking the reverse–complement. In this section, we enumerate $B_n^{180}(R)$ for each $R \subseteq B_2$.

We start by enumerating the permutations invariant under R_{180} with no pattern avoidance.

Theorem 2.1 *For all $k \geq 0$,*

$$|B_{2k}^{180}| = 4^k k!$$

and

$$|B_{2k+1}^{180}| = 2 \cdot |B_{2k}^{180}| = 2 \cdot 4^k k!.$$

We give two different proofs.

Direct Proof. Let $\pi \in B_{2k}^{180}$. There are $4k$ choices for the first column of the diagram of π: $2k$ positions and 2 signs. In addition, the first column's dot determines the last column's dot. Now there are only $4k - 4$ choices of what to put in the second column, and this determines the second-to-last column. We continue in this way until column k, in which only 4 choices remain. Thus, $|B_{2k}^{180}| = (4k)(4k-4)\cdots(8)(4) = 4^k k!$.

Now let $\pi \in B_{2k+1}^{180}$. There must be a dot in the central box, and there are two choices for its sign. Remove this dot's row and column; what remains is in B_{2k}^{180}. So $|B_{2k+1}^{180}| = 2 \cdot |B_{2k}^{180}| = 2 \cdot 4^k k!$. ■

Bijective Proof. A *four-colored permutation* is a permutation, written in one-line notation, in which every entry has a superscript chosen from the set $\{0, 1, 2, 3\}$. We can make diagrams of four-colored permutations in the same way as signed permutations, except that for each dot we choose a color from a set of four colors. Let Q_n be the set of four-colored permutations of length n. Since we have 4 choices of color for each entry, $|Q_n| = 4^n n!$.

We construct a bijection from Q_n to B_{2n}^{180} as follows. Given $\pi \in Q_n$, copy the $n \times n$ diagram of π into the lower-left quadrant of a $2n \times 2n$ grid. Reflect all of the color-2 and color-3 dots across the grid's horizontal center line, moving them into the upper-left quadrant. Then fill in the right half of the $2n \times 2n$ grid with a 180-degree rotation of the left half. Now replace all color-2 dots with color-0, and replace all color-3 dots with color-1. It is easy to check that this map sends π to a signed permutation in B_{2n}^{180}. The process is invertible, so it is a bijection. Therefore, $|B_{2k}^{180}| = |Q_k| = 4^k k!$.

The formula for $|B_{2k+1}^{180}|$ can now be proved in the same way as in the direct proof. ■

One thing that may be interesting to note is that B_{2k}^{180} is a group under the usual multiplication of signed permutations. Specifically, it is isomorphic to the semidirect product $V^k \rtimes S_k$, where V is the Klein four-group. To see this, we interpret an element π of B_{2k}^{180} as having two parts. The first part, which is an element of V^k, encodes whether each dot on the left half of π is barred or nonbarred, and whether it is in the top or bottom half of the permutation. The second part is an unsigned permutation of k, which encodes the unsigned permutation of the left half of π, taking the reflection over the horizontal line for any dots in the top half. The right half is then the image of the left half under R_{180}. For example, if $\pi = \bar{2}61\bar{5}\bar{4}83\bar{7} \in B_8^{180}$, then if a represents a barred dot in the lower half and b represents a nonbarred dot in the upper half, then π is represented as $((a, b, e, ab), 2314)$. Furthermore, this group is a subgroup of $B_{2k}^{180} \cup B_{2k}^{\overline{180}}$, which when interpreted in the same way is isomorphic to the group $(\mathbb{Z}_2^3)^k \rtimes S_k$. B_{2k}^{180} is a normal subgroup of index 2 in $B_{2k}^{180} \cup B_{2k}^{\overline{180}}$ since B_{2k}^{180} and $B_{2k}^{\overline{180}}$ are equinumerous. Note that B_{2k+1}^{180} is a group in the same way that B_{2k}^{180} is a group except that the choice of color for the middle square means that B_{2k+1}^{180} is isomorphic to $\mathbb{Z}_2 \oplus (V^k \rtimes S_k)$. Unfortunately, once we introduce pattern avoidance, the group structure disappears in general. Only in a few cases are the sets $B_{2k}^{180}(R)$ actually groups, and always a relatively trivial group. For instance, $B_{2k}^{180}(\bar{2}\bar{1}, \bar{2}1, 2\bar{1}, 21)$ is isomorphic to $(\mathbb{Z}_2)^k$ because it contains the strictly increasing permutations invariant under R_{180}.

Now we move on to sets with pattern avoidance. We have three different formulas that enumerate $B_{2k}^{180}(\bar{2}\bar{1})$.

Theorem 2.2 *For all $k \geq 0$, let $a_k = |B_{2k}^{180}(\bar{2}\bar{1})|$. Then the following hold.*

(i) For all $k \geq 0$,

$$a_k = \sum_{j=0}^{k} \binom{k}{j}^2 2^j j!. \tag{2.1}$$

(ii) $a_0 = 1$ and, for all $k \geq 1$,

$$a_k = 2k a_{k-1} + \sum_{j=0}^{k-1} \frac{2^{k-j-1}(k-1)!}{j!} a_j. \tag{2.2}$$

(iii) $a_0 = 1$, $a_1 = 3$, and, for all $k \geq 2$,

$$a_k = (4k-1)a_{k-1} - 4(k-1)^2 a_{k-2}. \tag{2.3}$$

(iv) For all $k \geq 0$,

$$|B_{2k+1}^{180}(\bar{2}\bar{1})| = 2a_k. \tag{2.4}$$

Proof. *(i)* One way to prove that (2.1) holds is by directly counting the number of ways we can form an element $\pi \in B_{2k}^{180}(\bar{2}\bar{1})$, by choosing which rows and columns of the diagram of π have barred dots and then ordering the nonbarred dots. However, we will instead prove this result by constructing a bijection with a restricted set of three-colored permutations.

A *three-colored permutation* is a permutation, written in one-line notation, in which every entry has a superscript chosen from the set $\{0, 1, 2\}$. We can make diagrams of three-colored permutations in the same way we do for signed permutations, except that for each dot we choose a color from a set of three colors. Given a three-colored permutation π, we say that π *avoids* $2^1 1^1$ whenever π has no pair of color-1 entries in descending order. Let $T_n(2^1 1^1)$ be the set of three-colored permutations of length n that avoid $2^1 1^1$. By [11, Thm. 5.1], $T_n(2^1 1^1)$ is enumerated by the formula in (2.1).

We construct a bijection from $T_n(2^1 1^1)$ to $B_{2n}^{180}(\bar{2}\bar{1})$ as follows. Given $\pi \in T_n(2^1 1^1)$, copy the $n \times n$ diagram of π into the lower-left quadrant of a $2n \times 2n$ grid. Reflect all of the color-2 dots across the grid's horizontal center line, moving them into the upper-left quadrant. Then fill in the right half of the $2n \times 2n$ grid with a 180-degree rotation of the left half. Now replace all color-2 dots with color-0.

Note that this map is the same as that used in proving Theorem 2.1, if we treat a three-color permutation as a four-color permutation in which no entry has color 3. It is easy to check that this map sends π to a signed permutation in $B_{2n}^{180}(\bar{2}\bar{1})$. The reason the map is surjective is that no permutation in $B_{2n}^{180}(\bar{2}\bar{1})$ can have any barred dot in its upper-left and lower-right quadrants (else the dot would form a $\bar{2}\bar{1}$ with its image). The process is invertible, so it is a bijection. Therefore, $|B_{2k}^{180}(\bar{2}\bar{1})| = |T_k(2^1 1^1)|$, and (2.1) follows.

(ii) Let $\pi \in B_{2k}^{180}(\bar{2}\bar{1})$ for $k \geq 1$. We consider two cases.

Case I. π has nonbarred 1. Then, since π is invariant under R_{180}, it also has nonbarred $2k$. We can put the 1 in any column, of which there are $2k$ choices. This determines the placement of $2k$ as well. The rest of the dots must form a permutation in $B_{2(k-1)}^{180}(\bar{2}\bar{1})$, of which there are a_{k-1}. So, in this case, the number of choices for π is $2ka_{k-1}$.

Case II. π has $\bar{1}$. Then, since π is invariant under R_{180}, it also has $\overline{2k}$. Since π avoids $\bar{2}\bar{1}$, the $\bar{1}$ must be before the $\overline{2k}$. So $\bar{1}$ is in the left half. Let $0 \leq j \leq k-1$ such that the $\bar{1}$ is in column $k-j$. All dots to the left of $\bar{1}$ must be nonbarred, to avoid $\bar{2}\bar{1}$. Then there are $2k-2$ choices for the first column, $2k-4$ choices remaining for the second column, …, and 2j + 2 choices remaining for the last column before the $\bar{1}$. Multiplying these together gives us $\frac{2^{k-j-1}(k-1)!}{j!}$ choices for the columns to the left of the $\bar{1}$. We have now determined (by symmetry) the part to the right of the $\overline{2k}$. What remains, in the $2j$ columns between the $\bar{1}$ and the $\overline{2k}$, must form a permutation in $B_{2j}^{180}(\bar{2}\bar{1})$, of which there are a_j. So, for each j, the number of choices for π is $\frac{2^{k-j-1}(k-1)!}{j!}a_j$. Summing this over all j from 0 to $k-1$ and adding the result of case I (namely, $2ka_{k-1}$) gives equation (2.2), as required.

(iii) We prove that the sequences in *(ii)* and *(iii)* are equal by showing that they have the same exponential generating function.

First we cover the sequence in *(ii)*, as given by a_k in (2.2). Let $F(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k$, the exponential generating function for this sequence. Replace k with $k+1$ in (2.2), multiply by $x^k/k!$, and sum over all $k \geq 0$; the result is

$$F'(x) = 2xF'(x) + 2F(x) + F(x)\frac{1}{1-2x},$$

so that

$$(1-2x)F'(x) = \left(2 + \frac{1}{1-2x}\right) F(x).$$

The exponential generating function for the sequence in *(ii)* is the solution to the above differential equation with $F(0) = a_0 = 1$, so

$$F(x) = \frac{e^{\frac{x}{1-2x}}}{1-2x}. \tag{2.5}$$

Second, we cover the sequence in *(iii)*, as given by a_k in (2.3). Let $G(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k$ be the exponential generating function, this time for the sequence in *(iii)*. Replace k with $k+2$ in (2.3), multiply by $\frac{x^k}{k!}$, and sum over all $k \geq 0$ to find that

$$G''(x) = 4xG''(x) + 7G'(x) - 4x^2G''(x) - 12xG'(x) - 4G(x),$$

so that

$$(1-4x+4x^2)G''(x) = (7-12x)G'(x) - 4G(x).$$

The exponential generating function for the sequence in *(iii)* is the solution to the above differential equation with $G(0) = a_0 = 1$ and $G'(0) = a_1 = 3$, so

$$G(x) = \frac{e^{\frac{x}{1-2x}}}{1-2x}. \tag{2.6}$$

Now since $F(x) = G(x)$, the result follows.

(iv) Let $\pi \in B_{2k+1}^{180}(\bar{2}\bar{1})$. Then, since π is invariant under R_{180}, the central box of π has a dot in it, which can be nonbarred or barred. Remove the row and column of this dot; what remains is in $B_{2k}^{180}(\bar{2}\bar{1})$. To see

that this process is invertible (up to the choice of the central dot's sign), we first observe that inserting a nonbarred entry in the central position of a permutation in $B_{2k}^{180}(\bar{2}\bar{1})$ cannot create a $\bar{2}\bar{1}$ pattern. If insertion of a barred entry creates a $\bar{2}\bar{1}$ pattern, then by R_{180} invariance the other entry in this pattern and its image under R_{180} must also form a $\bar{2}\bar{1}$ pattern, contradicting the fact that we started with a permutation which avoids $\bar{2}\bar{1}$. Now the result follows. ■

Theorem 2.3 *For all $k \geq 0$,*

$$|B_{2k}^{180}(\bar{2}\bar{1}, \bar{1}\bar{2})| = 2^k k!$$

and

$$|B_{2k+1}^{180}(\bar{2}\bar{1}, \bar{1}\bar{2})| = 2^{k+1} k!.$$

Proof. If $\pi \in B_n^{180}(\bar{2}\bar{1}, \bar{1}\bar{2})$ (n can be even or odd) has more than one barred dot, then it must contain either a $\bar{1}\bar{2}$ or a $\bar{2}\bar{1}$. If the jth column has a barred dot, then so does the $(n-j+1)$th column, as the image of the jth-column dot under R_{180}. Thus, $j = n-j+1$, and so the barred dot (if there is one) can only be in the center square (when n is odd). If we remove the central row and column, then what remains is in S_{2k}^{180} (where $n = 2k$ or $n = 2k+1$), and so there are $2^k k!$ choices for this [8, Thm. 2.5]. The result follows. ■

Theorem 2.4 *For all $k \geq 0$,*

$$|B_{2k}^{180}(\bar{2}\bar{1}, 21)| = \binom{2k}{k}$$

and

$$|B_{2k+1}^{180}(\bar{2}\bar{1}, 21)| = 2\binom{2k}{k}.$$

Proof. Observe that $\pi \in B_n^{180}(\bar{2}\bar{1}, 21)$ if and only if the barred entries and the nonbarred entries of π form increasing sequences. We first prove the result when $n = 2k$. Consider the k rows in the lower half and the k columns in the left half of π. Choose any k of these $2k$ rows and columns. The chosen columns are for barred dots and the chosen rows are for nonbarred dots. By construction, there are the same number of barred rows and columns in the lower-left quadrant, so we fill them in in increasing order, fill in the nonbarred dots, and then complete the upper-right quadrant using R_{180}.

We now prove the result when $n = 2k+1$. If we remove the center row and column, we get a permutation with the next smallest even length. The center square's dot can be either barred or nonbarred, and so we have twice as many permutations here as in the next smallest even case. Every dot except the middle is in either the lower-left or upper-right quadrant, so this does not create a $\bar{2}\bar{1}$ or a 21. ■

The next set has the same formula as the previous one, and there is a natural bijection, as we will explain.

Theorem 2.5 *For all $n \geq 0$,*

$$|B_n^{180}(\bar{2}\bar{1}, 12)| = |B_n^{180}(\bar{2}\bar{1}, 21)|$$

(see Theorem 2.4).

Proof. We construct a bijection from $B_n^{180}(\bar{2}\bar{1}, 21)$ to $B_n^{180}(\bar{2}\bar{1}, 12)$. For each $\pi \in B_n^{180}(\bar{2}\bar{1}, 21)$, let π' be the result of reflecting the nonbarred dots of π across the central horizontal line. Observe that π' is a signed permutation since π is invariant under R_{180}, so j is nonbarred if and only if $n+1-j$ is nonbarred. Also, since π avoids $\bar{2}\bar{1}$ and 21, π' avoids $\bar{2}\bar{1}$ and 12, so $\pi' \in B_n^{180}(\bar{2}\bar{1}, 12)$. Since $\pi'' = \pi$, this map is a bijection, and the result follows. ■

In the following theorem, we give proofs of two different formulas for the same set. The first is in terms of another set of restricted permutations, and the second is a recurrence relation.

Theorem 2.6 *Let $a_k = |B_{2k}^{180}(\bar{2}1, 2\bar{1})|$. Then the following hold.*

(i) For all $k \geq 0$,

$$a_k = 2^k k! + \sum_{j=0}^{k-1} 2^{k-j}(k-j)!|B_j(\bar{2}1, 2\bar{1})|. \tag{2.7}$$

(ii) $a_0 = 1$, $a_1 = 4$, *and, for all* $k \geq 2$,

$$a_k = 2^{k+1}k! + \sum_{j=1}^{k-1}(k-j)!a_j. \tag{2.8}$$

(iii) For all $k \geq 0$,

$$|B^{180}_{2k+1}(\bar{2}1, 2\bar{1})| = a_k + |B_k(\bar{2}1, 2\bar{1})|. \tag{2.9}$$

Proof. *(i)* Let $\pi \in B^{180}_{2k}(\bar{2}1, 2\bar{1})$. Then, if π has either only barred entries or only nonbarred entries, then the patterns are automatically avoided, and so there are $2^k k!$ permutations for each of these cases.

If π has at least one barred dot and one nonbarred dot, then π must consist of smaller single-sign permutations centered on the D diagonal. These subpermutations alternate in sign; otherwise we can combine two of them. Since π is invariant under R_{180}, the subpermutations on the right half of π are the images under R_{180} of the subpermutations on the left half. Also, there is a middle subpermutation since the kth and $(k+1)$th elements of π must have the same sign. This middle subpermutation must be invariant under R_{180}, and there is only one possible sign, the opposite sign from the adjacent permutations. Therefore, we choose $2j$ the size of the center subpermutation, choose the center subpermutation as an unsigned permutation invariant under R_{180}, and then choose the subpermutations on the left half by noting that the lower-left $(k-j)$-square is an element of $B_{k-j}(\bar{2}1, 2\bar{1})$. The formula follows.

(ii) The initial terms a_0 and a_1 are easy to verify.

Define $A_{k,j}$ to be the set of all $\pi \in B^{180}_{2k}(\bar{2}1, 2\bar{1})$ such that j is the length of the longest single-sign prefix of π. For example, $213\bar{5}\bar{4}687 \in A_{4,3}$. If $\pi \in B^{180}_{2k}(\bar{2}1, 2\bar{1})$ is all of the same sign, then we will say $\pi \in A_{k,k}$. Lastly, let $a_{k,j} = |A_{k,j}|$.

Let $\pi \in A_{k,k}$ for $k \geq 1$. Since π is all the same sign, the pattern restrictions do not apply: π can be any permutation in S^{180}_{2k}, either all nonbarred or all barred. Since $|S^{180}_{2k}| = 2^k k!$ by [8, Thm. 2.5], for all $k \geq 1$, we have

$$a_{k,k} = 2 \cdot |S^{180}_{2k}| = 2^{k+1}k!.$$

Now let $1 \leq j \leq k-1$ and $\pi \in A_{k,j}$. Without loss of generality, π begins with a nonbarred entry. We can add 1 to every entry in π and insert 1 and $2k+2$ (barred or nonbarred). Since π is invariant under R_{180}, we must place 1 and $2k+2$ opposite from each other in π; we now examine how to insert them. If we insert 1 (nonbarred) into π, it must go somewhere in the nonbarred prefix in order to avoid $\bar{2}1$; there are $j+1$ places for this, and each one results in a unique $\pi' \in A_{k+1,j+1}$. If we insert $\bar{1}$, it must go in the first position in order to avoid $2\bar{1}$; this results in one $\pi' \in A_{k+1,1}$. Thus, if we extend π to π' by inserting 1 and $2k+2$ (barred or nonbarred), then there are $j+1$ choices such that $\pi' \in A_{k+1,j+1}$ and one choice such that $\pi' \in A_{k+1,1}$. (Note that, if $\pi \in A_{k,k}$, then there is one way for it to extend to a $\pi' \in A_{k+1,1}$.)

Since this insertion process is invertible (up to choice of insertion point), it yields the following recurrence relation:

$$a_{k,k} = 2^{k+1}k! \qquad (k \geq 1); \tag{2.10}$$

$$a_{k,1} = \sum_{j=1}^{k-1} a_{k-1,j} = a_{k-1} \qquad (k \geq 2); \tag{2.11}$$

$$a_{k,j} = j a_{k-1,j-1} \qquad (2 \leq j \leq k-1). \tag{2.12}$$

It follows inductively from equations (2.11) and (2.12) that, for all j with $1 \leq j \leq k-1$,

$$a_{k,j} = j! a_{k-j+1,1} = j! a_{k-j}. \tag{2.13}$$

Now, since $a_k = \sum_{j=1}^{k} a_{k,j}$, the recurrence in (2.8) follows from (2.10) and (2.13).

(iii) We construct a bijection from $B^{180}_{2k+1}(\bar{2}1, 2\bar{1})$ to the union $B^{180}_{2k}(\bar{2}1, 2\bar{1}) \cup B_k(\bar{2}1, 2\bar{1})$, as follows. Let $\pi \in B^{180}_{2k+1}(\bar{2}1, 2\bar{1})$. If the center dot is the same sign as the dots in its adjacent columns, then remove the center row and column; the result is a unique element of $B^{180}_{2k}(\bar{2}1, 2\bar{1})$. Else, all the non-central dots in π

must be in the lower-left and upper-right quadrants, because of the pattern restrictions; the lower-left $k \times k$ quadrant is a unique element of $B_k(\bar{2}1, 2\bar{1})$. It is easy to check that this map is a bijection, and the result follows. ■

We remark that the set $B_n(\bar{2}1, 2\bar{1})$, which is used in the preceding theorem, is enumerated by [14, Thm. 2.2] and by our Lemma 3.5.

Theorem 2.7 *For all $n \geq 0$,*

$$|B_n^{180}(\bar{2}\bar{1}, \bar{1}2, 1\bar{2})| = 2^{\lfloor n/2 \rfloor} \lfloor n/2 \rfloor! + \sum_{j=0}^{\lceil n/2 \rceil - 1} j!.$$

Proof. Let $\pi \in B_n^{180}(\bar{2}\bar{1}, \bar{1}2, 1\bar{2})$. In the case where π has no barred dots, π is any unsigned permutation invariant under R_{180}, of which there are $2^{\lfloor n/2 \rfloor} \lfloor n/2 \rfloor!$ (see [8, Thm. 2.5]). In the case where π has at least one barred dot, let j be the number of nonbarred dots before the first barred dot of π. Because of the pattern restrictions on π, the barred dots must all be in the center and increasing, and the nonbarred dots must be in the upper-left and lower-right corners. Choose an unsigned permutation for the upper-left corner, of which there are $j!$. So there are $\sum_{j=0}^{\lceil n/2 \rceil - 1} j!$ choices in all, and the result follows from adding the two cases together. ■

Theorem 2.8 *Let $a_k = |B_{2k}^{180}(\bar{2}\bar{1}, 2\bar{1}, \bar{2}1)|$. Then $a_0 = 1$ and, for all $k \geq 1$,*

$$a_k = 2^k k! + \sum_{j=0}^{k-1} (k - j - 1)! a_j.$$

Furthermore, let $b_k = |B_{2k+1}^{180}(\bar{2}\bar{1}, 2\bar{1}, \bar{2}1)|$. Then $b_0 = 2$ and, for all $k \geq 1$,

$$b_k = (2^k + 1)k! + \sum_{j=0}^{k-1} (k - j - 1)! b_j.$$

Proof. We first prove the recurrence relation for a_k, the even-length enumeration. The initial term a_0 is easy to verify.

Define $A_{k,j}$ to be the set of all $\pi \in B_{2k}^{180}(\bar{2}\bar{1}, 2\bar{1}, \bar{2}1)$ such that the first barred entry is in position j. If $\pi \in B_{2k}^{180}(\bar{2}\bar{1}, 2\bar{1}, \bar{2}1)$ is entirely nonbarred, then we will say $\pi \in A_{k,k+1}$. Lastly, let $a_{k,j} = |A_{k,j}|$.

Let $\pi \in A_{k,k+1}$ for $k \geq 1$. Since π is all nonbarred, the pattern restrictions do not apply: π can be any permutation in S_{2k}^{180}, without bars. Since $|S_{2k}^{180}| = 2^k k!$ [8, Thm. 2.5], we have $a_{k,k+1} = 2^k k!$ for all $k \geq 1$.

Now let $1 \leq j \leq k$ and $\pi \in A_{k,j}$. We can add 1 to every entry in π and insert 1 and $2k + 2$ (barred or nonbarred). Since π is invariant under R_{180}, we must place 1 and $2k + 2$ opposite from each other in π; we now examine how to insert them. If we insert $2k + 2$ (nonbarred) into π, it must go before the first barred entry in order to avoid $2\bar{1}$; there are j places for this, and each one results in a unique $\pi' \in A_{k+1,j+1}$. If we insert $\overline{2k+2}$, it must go in the last position in order to avoid $\bar{2}1$ and $\bar{2}\bar{1}$, and so $\bar{1}$ goes in the first position; this results in one $\pi' \in A_{k+1,1}$. Thus, if we extend π to π' by inserting 1 and $2k + 2$ (barred or nonbarred), then there are j choices such that $\pi' \in A_{k+1,j+1}$ and 1 choice such that $\pi' \in A_{k+1,1}$. (Note that, if $\pi \in A_{k,k+1}$, then there is one way for it to extend to a $\pi' \in A_{k+1,1}$.)

Since this insertion process is invertible (up to choice of insertion point), it yields the following recurrence relation:

$$a_{k,k+1} = 2^k k! \qquad (k \geq 1); \tag{2.14}$$

$$a_{k,1} = \sum_{j=1}^{k} a_{k-1,j} = a_{k-1} \qquad (k \geq 1); \tag{2.15}$$

$$a_{k,j} = (j - 1) a_{k-1,j-1} \qquad (2 \leq j \leq k). \tag{2.16}$$

It follows inductively from equations (2.15) and (2.16) that, for all j with $1 \leq j \leq k$,

$$a_{k,j} = (j - 1)! a_{k-j+1,1} = (j - 1)! a_{k-j}. \tag{2.17}$$

Now, since $a_k = \sum_{j=1}^{k+1} a_{k,j}$, the required recurrence for a_k (the even-length enumeration) follows from (2.14) and (2.17).

The recurrence relation required for b_k, the odd-length enumeration, can be proved in the same way. The subtle difference in the formula ($2^k + 1$ instead of 2^k) comes from the fact that the first barred entry is now allowed to be placed in the $(k+1)$th position, in the center of the diagram. ■

Theorem 2.9 *For all $n \geq 1$,*

$$|B_n^{180}(\bar{2}1, \bar{1}2, 1\bar{2}, 2\bar{1})| = 2 \cdot |S_n^{180}| = 2^{\lfloor n/2 \rfloor + 1} \lfloor n/2 \rfloor !.$$

Proof. Let $\pi \in B_n^{180}(\bar{2}1, \bar{1}2, 1\bar{2}, 2\bar{1})$. Our avoidances mean that the entire permutation must have the same sign. Once we have chosen the sign, we just fill in an unsigned permutation invariant under R_{180}. The formula for the latter is given by [8, Thm. 2.5], and our formula here follows. ■

Theorem 2.10 *For all $k \geq 0$,*

$$|B_{2k}^{180}(\bar{2}\bar{1}, \bar{2}1, \bar{1}\bar{2}, 2\bar{1})| = 2^k k!$$

and

$$|B_{2k+1}^{180}(\bar{2}\bar{1}, \bar{2}1, \bar{1}\bar{2}, 2\bar{1})| = (2^k + 1)k!.$$

Proof. First note that if $\pi \in B_n^{180}$ avoids $\bar{2}\bar{1}$ and $\bar{1}\bar{2}$, then π has at most one barred entry. If n is even, then there are no barred entries, so we just have an element of S_n^{180}, which is enumerated in [8, Thm. 2.5].

If n is odd, then the center square has either a nonbarred dot or a barred dot. In the former case, the rest of the permutation is an element of S_{2k}^{180} (where $n = 2k + 1$), which is enumerated in [8, Thm. 2.5]. In the latter case, everything on the left must be below the center row, so we can choose any signed permutation in the bottom left quadrant—for which there are $k!$ choices. Since π is invariant under R_{180}, the right half and thus the entirety of π are now determined. ■

The rest of the results in this section are more straightforward, so we leave their proofs as exercises to the reader. They are listed in Table 1, along with the other results from this section.

n	R	$\lvert B_n^{180}(R)\rvert$	Recurrence
$2k$ $2k+1$	$\emptyset$	$4^k k!$ $2 \cdot 4^k k!$	
$2k$ $2k+1$	$\{\bar{2}\bar{1}\}$	$\sum_{j=0}^{k} \binom{k}{j}^2 2^j j!$ $2\sum_{j=0}^{k} \binom{k}{j}^2 2^j j!$	$a_k = (4k-1)a_{k-1} - 4(k-1)^2 a_{k-2}$ $b_k = 2a_k$
$2k$ $2k+1$	$\{\bar{2}\bar{1}, \bar{1}\bar{2}\}$	$2^k k!$ $2^{k+1} k!$	
$2k$ $2k+1$	$\{\bar{2}\bar{1}, 21\}$	$\binom{2k}{k}$ $2\binom{2k}{k}$	
$2k$ $2k+1$	$\{\bar{2}\bar{1}, 12\}$	$\binom{2k}{k}$ $2\binom{2k}{k}$	
$2k$ $2k+1$	$\{\bar{2}1, 2\bar{1}\}$	Theorem 2.6*(i)*	$a_k = 2^{k+1}k! + \sum_{j=1}^{k-1}(k-j)!a_j$ $b_k = a_k + \lvert B_k(\bar{2}1, 2\bar{1})\rvert$
n	$\{\bar{2}\bar{1}, \bar{1}2, 1\bar{2}\}$	$2^{\lfloor n/2 \rfloor} \lfloor n/2 \rfloor! + \sum_{j=0}^{\lceil n/2 \rceil - 1} j!$	
$2k$ $2k+1$	$\{\bar{2}\bar{1}, 2\bar{1}, \bar{2}1\}$		$a_k = 2^k k! + \sum_{j=0}^{k-1}(k-j-1)!a_j$ $b_k = (2^k+1)k! + \sum_{j=0}^{k-1}(k-j-1)!b_j$
n	$\{\bar{2}1, \bar{1}2, 1\bar{2}, 2\bar{1}\}$	$2^{\lfloor n/2 \rfloor + 1} \lfloor n/2 \rfloor!$	
n	$\{\bar{2}\bar{1}, \bar{2}1, 2\bar{1}, 21\}$	$2^{\lceil n/2 \rceil}$	
$2k$ $2k+1$	$\{\bar{2}\bar{1}, \bar{2}1, \bar{1}\bar{2}, 2\bar{1}\}$	$2^k k!$ $(2^k+1)k!$	
n	$\{\bar{2}\bar{1}, \bar{2}1, \bar{1}2, 1\bar{2}, 2\bar{1}\}$	$1 + 2^{\lfloor n/2 \rfloor} \lfloor n/2 \rfloor!$	
n	$\{\bar{2}\bar{1}, \bar{2}1, \bar{1}\bar{2}, \bar{1}2, 1\bar{2}, 2\bar{1}\}$	$2^{\lfloor n/2 \rfloor} \lfloor n/2 \rfloor!$	
n	$\{\bar{1}\bar{2}, \bar{2}\bar{1}, 12, 21\}$	0	Note: the same is true for any $S \supseteq R$.
$2k$ $2k+1$	$\{\bar{2}\bar{1}, \bar{2}1, 12, 2\bar{1}\}$	$k+1$ $k+2$	
n	$\{\bar{2}\bar{1}, \bar{2}1, \bar{1}2, 1\bar{2}, 2\bar{1}, 21\}$	2	
n	$\{\bar{2}\bar{1}, \bar{2}1, \bar{1}\bar{2}, 12, 2\bar{1}\}$	1	
n	$\{\bar{2}\bar{1}, \bar{2}1, \bar{1}\bar{2}, \bar{1}2, 1\bar{2}, 12, 2\bar{1}\}$	1	
$2k$ $2k+1$	$\{\bar{2}\bar{1}, \bar{1}\bar{2}, 12\}$	1 2	
$2k$ $2k+1$	$\{\bar{2}\bar{1}, \bar{2}1, \bar{1}\bar{2}, 2\bar{1}, 21\}$	1 2	
n	$\{\bar{2}\bar{1}, \bar{1}2, 1\bar{2}, 21\}$	2	
n	$\{\bar{2}\bar{1}, \bar{2}1, \bar{1}2, 1\bar{2}, 2\bar{1}, 21\}$	2	
n	$\{\bar{2}\bar{1}, \bar{2}1, \bar{1}2, 1\bar{2}, 12, 2\bar{1}\}$	2	

Table 1: Enumerations of signed permutations of length at least 4 that are invariant under R_{180} and avoid R.

Every $R \subseteq B_2$ that has not been explicitly used as a pattern-avoidance set in this section is equivalent by Lemma 1.1 to one of the sets that has been. Thus, we have now enumerated $B_n^{180}(R)$ for every $R \subseteq B_2$.

3 Signed permutations invariant under $\overline{R_{180}}$

Let $B_n^{\overline{180}}$ be the set of signed permutations of length n that are invariant under the subgroup $\langle \overline{R_{180}} \rangle = \{e, \overline{R_{180}}\}$ of $\mathfrak{S}$. Recall from our introduction that applying $\overline{R_{180}}$ to a signed permutation is the same as taking the reverse–complement and applying the bar operation. In this section, we enumerate $B_n^{\overline{180}}(R)$ for each $R \subseteq B_2$.

A signed permutation of odd length cannot be invariant under $\overline{R_{180}}$, since nothing can be placed in the center column without violating the symmetry. Thus, $B_{2k+1}^{\overline{180}}(R) = \emptyset$ for each $R \subseteq B_2$. Now it remains only to enumerate $B_{2k}^{\overline{180}}(R)$ for each $R \subseteq B_2$.

We start by enumerating the permutations invariant under $\overline{R_{180}}$ with no pattern avoidance.

Theorem 3.1 *For all $k \geq 0$,*

$$|B_{2k}^{\overline{180}}| = |B_{2k}^{180}| = 4^k k!.$$

Proof. There is a bijection between B_{2k}^{180} and $B_{2k}^{\overline{180}}$, as follows: given a signed permutation in B_{2k}^{180}, change the sign on everything in the right half. The formula now follows from Theorem 2.1. ■

Now we move on to sets with pattern avoidance.

Theorem 3.2 *For all $k \geq 0$,*

$$|B_{2k}^{\overline{180}}(\bar{2}1)| = |B_{2k}^{180}(\bar{2}\bar{1})|,$$

the latter of which is enumerated in Theorem 2.2.

Proof. We construct a map as follows. Let $\pi \in B_{2k}^{180}(\bar{2}\bar{1})$. Look at the left half of π. Every place where a $\bar{2}1$ pattern occurs, put a bar on the second entry of it; this turns it into a $\bar{2}\bar{1}$. Then perform $\overline{R_{180}}$ on the new left half to get the new right half.

Now we must check that the resulting π' is an element of $B_{2k}^{\overline{180}}(\bar{2}1)$. It is clear from the construction of the map that π' is invariant under $\overline{R_{180}}$ and that π' has no $\bar{2}1$ patterns contained in its left half. Further, π' cannot have a $\bar{2}1$ pattern contained in its right half, because $\overline{R_{180}}$ would map it to a $\bar{2}1$ pattern in the left half.

Every barred dot in the left half of π must have been in the lower half, or else it would have formed a $\bar{2}\bar{1}$ with its image under R_{180}. In the left half of π', all of the new barred dots from the map were made lower than the barred dots that were already in π. Thus, every barred dot in the left half of π' is in the lower half. Then, since π' is invariant under $\overline{R_{180}}$, every nonbarred dot in the right half of π' is in the upper half. So π' cannot have a $\bar{2}1$ pattern anywhere. Therefore, the map goes into $B_{2k}^{\overline{180}}(\bar{2}1)$.

It is easy to verify that this map can be inverted, so it is a bijection between $B_{2k}^{\overline{180}}(\bar{2}1)$ and $B_{2k}^{180}(\bar{2}\bar{1})$. ■

Theorem 3.3 *For all $k \geq 0$,*

$$|B_{2k}^{\overline{180}}(\bar{2}\bar{1}, 21)| = 2^{2k}.$$

Proof. To construct $\pi \in B_{2k}^{\overline{180}}(\bar{2}\bar{1}, 21)$, for each $j \leq k$, choose whether column j or column $2k-j+1$ of π has a barred dot (exactly one of these columns has a barred dot since π is invariant under $\overline{R_{180}}$). Similarly, for each j, choose whether row j or row $2k-j+1$ has a barred dot. Then place the barred dots and nonbarred dots in ascending order in the appropriate spaces. ■

Theorem 3.4 *Let $a_k = |B_{2k}^{\overline{180}}(\bar{2}1, 2\bar{1})|$ for all $k \geq 0$. Then $a_0 = 1$ and, for all $k \geq 1$,*

$$a_k = k! + \sum_{j=0}^{k-1} (k-j)! a_j.$$

Before we can prove this theorem, we need to enumerate $B_n(\bar{2}1, 2\bar{1})$. The reverse map is a bijection to $B_n(1\bar{2}, \bar{1}2)$, which Mansour and West [14, Thm. 2.2] enumerate with an explicit formula involving a sum over compositions of n:

$$|B_n(1\bar{2}, \bar{1}2)| = 2 \sum_{\ell=1}^{n} \sum_{\substack{i_1+i_2+\cdots+i_\ell=n \\ i_j \geq 1}} \prod_{j=1}^{\ell} i_j!. \tag{3.1}$$

Here we give a recurrence relation for $|B_n(1\bar{2}, \bar{1}2)|$ that is simpler than the formula above.

Lemma 3.5 *Let $a_n = |B_n(1\bar{2}, \bar{1}2)|$. Then $a_0 = 1$ and, for all $n \geq 1$,*

$$a_n = n! + \sum_{j=0}^{n-1} (n-j)! a_j.$$

Proof. The $n = 0$ case is easily verified. Define $A_{n,j}$ to be the set of all $\pi \in B_n(1\bar{2}, \bar{1}2)$ such that j is the length of the longest single-sign prefix of π. For example, $756\bar{4}\bar{3}\bar{2}\bar{1} \in A_{7,3}$. Let $a_{n,j} = |A_{n,j}|$.

Let $1 \leq j \leq n$ and $\pi \in A_{n,j}$. Without loss of generality, suppose π begins with a nonbarred entry. If we insert $n+1$ into π, it must go somewhere in the nonbarred prefix in order to avoid $\bar{1}2$; there are $j+1$ places for this, and each one results in a unique $\pi' \in A_{n+1,j+1}$. If we insert $\overline{n+1}$, it must go in the first position in order to avoid $1\bar{2}$; this results in one $\pi' \in A_{n+1,1}$. Thus, if we extend π to π' by inserting a $n+1$ or $\overline{n+1}$, then there are $j+1$ choices such that $\pi' \in A_{n+1,j+1}$ and 1 choice such that $\pi' \in A_{n+1,1}$.

Since this insertion process is invertible (up to choice of insertion point), it yields the following recurrence relation:

$$a_{1,1} = 2; \tag{3.2}$$

$$a_{n,1} = \sum_{j=1}^{n-1} a_{n-1,j} = a_{n-1} \qquad (n \geq 2); \tag{3.3}$$

$$a_{n,j} = j a_{n-1,j-1} \qquad (2 \leq j \leq n). \tag{3.4}$$

It follows inductively from equation (3.4) that, for all j with $1 \leq j \leq n$,

$$a_{n,j} = j! a_{n-j+1,1}. \tag{3.5}$$

Now, since $a_n = \sum_{j=1}^{n} a_{n,j}$, the required recurrence for a_n follows from (3.2), (3.3), and (3.5). ∎

We now prove Theorem 3.4 using Lemma 3.5.

Proof of Theorem 3.4. Let $\pi \in B_{2k}^{\overline{180}}(\bar{2}1, 2\bar{1})$. If there is anything in the upper-left quadrant, it is higher than its image under $\overline{R_{180}}$, which will create a $\bar{2}1$ or a $2\bar{1}$. Therefore, fill in the lower-left quadrant with a permutation avoiding $\bar{2}1$ and $2\bar{1}$, and this determines the right half. This is a bijection between $B_{2k}^{\overline{180}}(\bar{2}1, 2\bar{1})$ and $B_k(\bar{2}1, 2\bar{1})$. The reverse map is a bijection between $B_k(\bar{2}1, 2\bar{1})$ and $B_k(1\bar{2}, \bar{1}2)$. Compose these bijections, and the required recurrence relation follows from Lemma 3.5. ∎

Theorem 3.6 *For all $k \geq 0$,*

$$|B_{2k}^{\overline{180}}(\bar{2}1, \bar{1}2)| = 2^k k!.$$

Proof. Note that $\pi \in B_{2k}^{\overline{180}}$ avoids $\bar{2}1$ and $\bar{1}2$ if and only if π has only nonbarred dots in its left half and only barred dots in its right half. Therefore, we can uniquely construct each element of $B_{2k}^{\overline{180}}(\bar{2}1, \bar{1}2)$ by choosing an element of S_{2k}^{180} (see [8, Thm. 2.5]) and putting bars over the dots in its right half. ∎

Theorem 3.7 *For all $k \geq 0$,*

$$|B_{2k}^{\overline{180}}(\bar{2}1, \bar{1}2, 1\bar{2})| = k!.$$

Proof. For every $\pi \in B_{2k}^{\overline{180}}(\bar{2}1, \bar{1}2, 1\bar{2})$, the nonbarred dots of π must be in the left half and the barred dots must be in the right half. Thus, the nonbarred dots must be in the upper half and the barred dots must be in the lower half. Choose a nonbarred permutation for the upper-left quadrant; this determines the entire permutation. ∎

Theorem 3.8 *For all $k \geq 0$,*

$$|B_{2k}^{\overline{180}}(\bar{2}\bar{1}, \bar{1}2, 21)| = 2^{k+1} - 1.$$

Proof. Let $\pi \in B_{2k}^{\overline{180}}(\bar{2}\bar{1}, \bar{1}2, 21)$. Every dot in the lower-left quadrant of π must be nonbarred, or else it forms a $\bar{1}2$ with its image under $\overline{R_{180}}$. For the same reason, every dot in the upper-right quadrant must be barred. We now look at what can go in the upper-left quadrant.

Suppose the upper-left quadrant of π consists of j barred dots. They must be placed in the bottommost j rows of the upper-left quadrant, or else they form a $\bar{2}\bar{1}$ with something in the upper-right quadrant. The upper-left quadrant's barred dots must be in increasing order, to avoid $\bar{2}\bar{1}$. So, since we need only to pick the columns to put them in, there are $\binom{k}{j}$ possible configurations with j barred dots in the upper-left quadrant. This choice of columns also determines the lower-left quadrant, and thus the entirety of π. Therefore, in the case where the upper-left quadrant consists of any positive number of barred dots, the number of choices for π is

$$\sum_{j=1}^{k} \binom{k}{j} = 2^k - 1.$$

Now suppose the upper-left quadrant of π consists of j nonbarred dots. By the same reasoning as above, they must be placed in increasing order in the rightmost columns of the upper-left quadrant, and the number of choices for π in this case is $2^k - 1$ again.

The upper-left quadrant cannot contain both a barred dot and a nonbarred dot, or else there would be a $\bar{1}2$ formed by the barred dot in the quadrant's bottommost row and the nonbarred dot in the quadrant's rightmost column. There is only one other choice for π: putting no dots in the upper-left quadrant. Thus, the total number of possible π is $(2^k - 1) + (2^k - 1) + 1 = 2^{k+1} - 1$. ■

Theorem 3.9 *For all $k \geq 0$,*

$$|B_{2k}^{\overline{180}}(\bar{2}\bar{1}, \bar{2}1, 21)| = \binom{2k+1}{k}.$$

Proof. Our proof is similar to Egge's proof that $|S_{2k}^{180}(123)| = \binom{2k}{k}$ [8, Thm. 2.17]. We compute the generating tree for constructing the permutations in $B_{2k}^{\overline{180}}(\bar{2}\bar{1}, \bar{2}1, 21)$, and then we use the kernel method to find the generating function for $\{|B_{2k}^{\overline{180}}(\bar{2}\bar{1}, \bar{2}1, 21)|\}_{k=0}^{\infty}$. (For more about generating trees, see [20].) On each branch, we add the bottom and top rows into the permutation (they are images under $\overline{R_{180}}$). "Adding" a permutation involves placing a dot in the bottom row and another dot at its image in the top row. For example, to add $\bar{1}2$, we place a barred dot in the left half of the bottom row and a nonbarred dot at the barred dot's image in the right half of the top row. We can add the following permutations:

- $2\bar{1}$ iff the right half consists completely of barred dots; we add this into the middle two columns;
- $1\bar{2}$ in any situation; we put the 1 in the leftmost column;
- $\bar{1}2$ in any situation; we put the $\bar{1}$ anywhere to the left of all barred dots.

We call a permutation a *brahmin* whenever it has no barred dot in the left half. We label each non-brahmin with j, where $j-2$ is the number of nonbarred dots to the left of the first barred dot; then, a permutation with label j has j children, all non-brahmins, with labels $2, 3, \ldots, j+1$. Each brahmin has two brahmin children and k non-brahmin children, with labels $2, 3, \ldots, k+1$. Note that this means that there are 2^k brahmins of length $2k$.

We can count the non-brahmins by k and j:

$$F(x, y) = \sum x^k y^j,$$

where the sum is over the non-brahmins in $\bigcup_{k \geq 0} B_{2k}^{\overline{180}}(\bar{2}\bar{1}, \bar{2}1, 21)$. Thus the generating function for $\{|B_{2k}^{\overline{180}}(\bar{2}\bar{1}, \bar{2}1, 21)|\}_{k=0}^{\infty}$ is $F(x, 1) + \frac{1}{1-2x}$.

Counting the non-brahmins by their parents, we get

$$\begin{aligned} F(x, y) &= \sum_{k=0}^{\infty} 2^k x^{k+1}(y^2 + \ldots + y^{k+1}) + x \sum_{j=2}^{\infty} F_j(x)(y^{j+1} + y^j + \ldots y^2) \\ &= \frac{xy^2}{y-1}\left(\frac{y}{1-2xy} - \frac{1}{1-2x}\right) + \frac{xy^2}{y-1}[F(x, y) - F(x, 1)], \end{aligned} \tag{3.6}$$

where $F_j(x)$ is the unique function for which

$$F(x, y) = \sum_{j=2}^{\infty} F_j(x) y^j.$$

($F_j(x)$ turns out to be unimportant in our proof.)

Setting $y = \frac{1-\sqrt{1-4x}}{2x}$ in (3.6), we have $\frac{xy^2}{y-1} = 1$ and

$$F(x,1) + \frac{1}{1-2x} = \frac{1-\sqrt{1-4x}}{2x\sqrt{1-4x}},$$

which is the ordinary generating function for $\binom{2n+1}{n}$. ■

Theorem 3.10 *For all $k \geq 0$,*

$$|B_{2k}^{\overline{180}}(\bar{2}\bar{1}, \bar{2}1, 2\bar{1}, 21)| = 2^k.$$

Proof. $\pi \in B_{2k}^{\overline{180}}(\bar{2}\bar{1}, \bar{2}1, 2\bar{1}, 21)$ must be strictly increasing, but we can choose the sign for each column on the left side, which determines the signs for the columns on the right side. ■

Theorem 3.11 *For all $k \geq 0$,*

$$|B_{2k}^{\overline{180}}(\bar{2}\bar{1}, \bar{2}1, \bar{1}2, 21)| = 2^k.$$

Proof. For $\pi \in B_{2k}^{\overline{180}}(\bar{2}\bar{1}, \bar{2}1, \bar{1}2, 21)$, the barred dots must be increasing, as must be the nonbarred dots. The nonbarred dots must be on the left half of π and the barred dots must be on the right half. Choose which of the rows in the lower half have nonbarred dots. Fill these in in the leftmost columns. Then fill in their images under $\overline{R_{180}}$; these are barred dots in the upper-right quadrant. Then, all of the remaining upper rows must be nonbarred dots in the left half; fill these in in increasing order. Finally, fill in these images, which are barred dots in the lower-right quadrant. ■

The rest of the results in this section are more straightforward, so we leave their proofs as exercises to the reader. They are listed in Table 2, along with the other results from this section.

n	R	$\lvert B_n^{\overline{180}}(R)\rvert$	Recurrence
$2k$	$\emptyset$	$4^k k!$	
$2k$	$\{\bar{2}1\}$	$\sum_{j=0}^{k} \binom{k}{j}^2 2^j j!$	$a_k = (4k-1)a_{k-1} - 4(k-1)^2 a_{k-2}$
$2k$	$\{\bar{2}\bar{1}, 21\}$	2^{2k}	
$2k$	$\{\bar{2}1, 2\bar{1}\}$		$a_k = k! + \sum_{j=0}^{k-1}(k-j)! a_j$
$2k$	$\{\bar{2}1, \bar{1}2\}$	$2^k k!$	
$2k$	$\{\bar{2}1, \bar{1}2, 1\bar{2}\}$	$k!$	
$2k$	$\{\bar{2}\bar{1}, \bar{1}2, 21\}$	$2^{k+1}-1$	
$2k$	$\{\bar{2}\bar{1}, \bar{2}1, 21\}$	$\binom{2k+1}{k}$	
$2k$	$\{\bar{2}\bar{1}, \bar{2}1, 2\bar{1}, 21\}$	2^k	
$2k$	$\{\bar{2}\bar{1}, \bar{2}1, \bar{1}2, 21\}$	2^k	
$2k$	$\{\bar{1}\bar{2}, \bar{2}\bar{1}, 12, 21\}$	0	Note: the same is true for any $S \supseteq R$.
$2k$	$\{\bar{2}1, \bar{1}2, 1\bar{2}, 2\bar{1}\}$	0	Note: the same is true for any $S \supseteq R$.
$2k$	$\{\bar{2}\bar{1}, \bar{1}2, 1\bar{2}, 21\}$	2	
$2k$	$\{\bar{2}\bar{1}, \bar{2}1, \bar{1}2, 1\bar{2}, 21\}$	1	
$2k$	$\{\bar{2}\bar{1}, \bar{2}1, \bar{1}2, 2\bar{1}, 21\}$	1	

Table 2: Enumerations of signed permutations of length at least 4 that are invariant under $\overline{R_{180}}$ and avoid R.

Every $R \subseteq B_2$ that has not been explicitly used as a pattern-avoidance set in this section is equivalent by Lemma 1.1 to one of the sets that has been. Thus, we have now enumerated $B_n^{\overline{180}}(R)$ for every $R \subseteq B_2$.

4 Signed permutations invariant under D and D'

For this section, let H be the subgroup $\{e, R_{180}, D, D'\}$ of $\mathfrak{S}$. Recall from our introduction that applying R_{180} is the same as taking the reverse–complement; D is the same as reflecting across the "/" diagonal (the D diagonal), or the same as taking the group-theoretic inverse; and $D' = D \circ R_{180} = R_{180} \circ D$ is the same as reflecting across the "\" diagonal (the D' diagonal).

We start by enumerating the permutations invariant under H with no pattern avoidance.

Theorem 4.1 *$|B_0^H| = 1$, $|B_2^H| = 4$, and, for all $k \geq 2$,*

$$|B_{2k}^H| = 4 \cdot |B_{2(k-1)}^H| + 4(k-1)|B_{2(k-2)}^H|.$$

Furthermore, for all $k \geq 0$,

$$|B_{2k+1}| = 2 \cdot |B_{2k}^H|.$$

Proof. The base cases are easy to verify. Now let $\pi \in B_{2k}^H$ with $k \geq 2$. We examine two cases.

Case I. If the first-column dot is in the top or bottom box, then there are 4 choices for it (2 position choices and 2 signs), and it determines only one other dot. Upon removing the row and column of each of these two dots, what remains is in $B_{2(k-1)}^H$. Thus, in this case, there are $4 \cdot |B_{2(k-1)}^H|$ permutations.

Case II. If the first-column dot is not in the top or bottom box, then there are $2(k-1)$ choices for the first-column dot's position and 2 for its sign, for a total of $4(k-1)$ choices. This dot determines three other dots, by the action of H. Upon removing the row and column of each of these four dots, what remains is in $B_{2(k-2)}^H$. Thus, in this case, there are $4(k-1)|B_{2(k-2)}^H|$ permutations. The sum of the results over both cases is the required recurrence relation for $|B_{2k}^H|$, the evens.

Now suppose that $\pi \in B_{2k+1}^H$. There must be a dot in the central box, for which there 2 sign choices. Upon removing its row and column, what remains is in $|B_{2k}^H|$. Therefore, $|B_{2k+1}^H| = 2 \cdot |B_{2k}^H|$. ■

Now we move on to sets with pattern avoidance.

Theorem 4.2 *$|B_0^H(\bar{2}\bar{1})| = 1$, $|B_1^H(\bar{2}\bar{1})| = 2$, $|B_2^H(\bar{2}\bar{1})| = 3$, $|B_3^H(\bar{2}\bar{1})| = 6$, and, for all $n \geq 4$,*

$$|B_n^H(\bar{2}\bar{1})| = 3|B_{n-2}^H(\bar{2}\bar{1})| + (n-2)|B_{n-4}^H(\bar{2}\bar{1})|.$$

Proof. We show this by induction. The base cases are left as an exercise to the reader. For the inductive case, for $n \geq 4$, choose the leftmost dot of $\pi \in B_n^H(\bar{2}\bar{1})$. If the dot is barred, it must be on the D diagonal of π because otherwise it, along with its image under D, would create a $\bar{2}\bar{1}$. In this case, remove the rows and columns of the leftmost dot and its image under D' (the rightmost dot), as they cannot be part of a $\bar{2}\bar{1}$, and proceed to the size-$(n-2)$ case. If the leftmost dot is nonbarred, it can be anywhere in the left column. In any case, the nonbarred dots cannot be a part of a $\bar{2}\bar{1}$, so we can remove the rows and columns of the leftmost dot and its images under the elements of H. If the leftmost dot is at the top or the bottom, this is a total of 2 dots; otherwise, it is a total of 4 dots. Adding all this up, we have shown the recurrence relation. ■

Theorem 4.3 *For all $n \geq 0$,*

$$|B_n^H(\bar{2}\bar{1}, 21)| = 2^{\lceil n/2 \rceil}.$$

Proof. Let $\pi \in B_n^H(\bar{2}\bar{1}, 21)$. If there were a dot of π not on the D diagonal, then it would form a $\bar{2}\bar{1}$ or a 21 with its image under D, and these are forbidden; thus, all dots in π lie on the D diagonal. So choose the sign of each dot in the left half, and the right half is determined. ■

The next set has the same formula as the previous one, and there is a natural bijection, as we will explain.

Theorem 4.4 *For all $n \geq 0$,*

$$|B_n^H(\bar{2}\bar{1}, 12)| = |B_n^H(\bar{2}\bar{1}, 21)| = 2^{\lceil n/2 \rceil}.$$

Proof. We construct a bijection from $B_n^H(\bar{2}\bar{1}, 21)$ to $B_n^H(\bar{2}\bar{1}, 12)$. Take $\pi \in B_n^H(\bar{2}\bar{1}, 21)$ and reflect the nonbarred dots across the central horizontal line to create π'. The reason π' is a permutation is that, if $\pi(j)$ is a nonbar, then $\pi(n-j)$ is also a nonbar, so π' will only have one entry in row $\pi(n-j)$. Also, since π avoids $\bar{2}\bar{1}$ and 21, π' avoids $\bar{2}\bar{1}$ and 12. Finally, π' is invariant under H. The map is an involution, so $|B_n^{180}(\bar{2}\bar{1}, 21)| = |B_n^{180}(\bar{2}\bar{1}, 12)|$. The formula now follows from Theorem 4.3. ■

Theorem 4.5 *Let $a_k = |B_{2k}^H(\bar{2}\bar{1}, \bar{1}\bar{2})|$. Then $a_0 = 1$, $a_1 = 2$, and, for all $k \geq 2$,*

$$a_k = 2a_{k-1} + 2(k-1)a_{k-2}.$$

Furthermore, for all $k \geq 0$,

$$|B_{2k+1}^H(\bar{2}\bar{1}, \bar{1}\bar{2})| = 2a_k.$$

Proof. The initial terms are easy to verify. We first prove the result for even-length permutations. Note that if $\pi \in B^H_{2k}(\bar{2}\bar{1}, \bar{1}\bar{2})$, then π has no barred dots. Choose the leftmost dot of π. If it is on either diagonal, then it has 2 images under H; otherwise it has 4. Remove the rows and columns of each image, and the recurrence relation follows.

We now prove the result for odd-length permutations. Note that the middle square of $\pi \in B^H_{2k+1}(\bar{2}\bar{1}, \bar{1}\bar{2})$ has either a nonbarred dot or a barred dot. Remove the middle row and column of π, and then the remaining permutation must be an element of $B^H_{2k}(\bar{2}\bar{1}, \bar{1}\bar{2})$. ■

We remark that, as seen in the preceding proof, the signed permutations in $B^H_{2k}(\bar{2}\bar{1}, \bar{1}\bar{2})$ are exactly the unsigned permutations in S^H_{2k}. That is, the two sets are equal. The latter is enumerated, in [8, Thm. 3.1], by the same recurrence relation that we prove here for the former.

Theorem 4.6 *Let $b_j = |B^D_j(\bar{2}1, 2\bar{1})|$, but with $b_0 = 2$, and let $c_j = |S^H_j|$, but with $c_1 = 2$. Then, for all $k \geq 1$,*

$$|B^H_{2k}(\bar{2}1, 2\bar{1})| = \sum_{j=0}^{k-1} b_j c_{2(k-j)},$$

and

$$|B^H_{2k+1}(\bar{2}1, 2\bar{1})| = \sum_{j=0}^{k} b_j c_{2(k-j)+1}.$$

Proof. To form an element π of $B^H_n(\bar{2}1, 2\bar{1})$, choose $j < \frac{n}{2}$. Fill in the lower-left square region of size j with a permutation avoiding $\bar{2}1$ and $2\bar{1}$ and invariant under D. This consists of smaller square regions of sizes $j_1, \ldots j_n$ each of which is a single-sign permutation invariant under D. Moving from left to right, these smaller single-signed permutations alternate in sign, from one permutation to the next. We can then fill in the upper-right square region of size j as the image of the lower-left square under D'. Finally, we fill in the remaining central square region of size $2k - 2j$ with a permutation of the sign opposite to the squares incident upon it that is invariant under D and D'. This uniquely determines π, and the formula follows. ■

We remark that the set S^H_n, used in the preceding theorem, is the set of unsigned permutations of n invariant under H, enumerated by [8, Thm. 3.1]. In particular, $|S^H_0| = 1$, $|S^H_2| = 2$, for all $k \geq 1$

$$|S^H_{2k}| = 2 \cdot |S^H_{2(k-1)}| + 2(k-1)|S^H_{2(k-2)}|, \tag{4.1}$$

and for all $k \geq 0$

$$|S^H_{2k+1}| = |S^H_{2k}|. \tag{4.2}$$

Theorem 4.7 *For all $n \geq 0$,*

$$|B^H_n(\bar{2}\bar{1}, \bar{1}2, 1\bar{2})| = |S^H_n| + \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} |S^D_j|.$$

Proof. Let $\pi \in B^H_n(\bar{2}\bar{1}, \bar{1}2, 1\bar{2})$. The number of possible π that have no barred dots is $|S^H_n|$, because none of the pattern restrictions affect nonbarred dots.

Now suppose that π has at least one barred dot. If there is a barred dot in π that is not on the D diagonal, then this dot and its image under D form a $\bar{2}\bar{1}$, a contradiction; thus, every barred dot in π is on the D diagonal. Since π avoids $\bar{1}2$, there cannot be a nonbarred dot anywhere above and to the right of the first barred dot. Since π avoids $1\bar{2}$, there cannot be a nonbarred dot anywhere below and to the left of the last barred dot. The only remaining places to put nonbarred dots are in two square regions in the upper-left and lower-right; the upper-left region consists of the boxes higher than the highest barred dot and to the left of the leftmost barred dot, and the lower-right region is the upper-left region's image under D. Given that the width of the upper-left region of nonbarred dots is j, the number and placement of the barred dots is uniquely determined. The upper-left region is an element of $S^{D'}_j$, so the number of choices for it is $|S^{D'}_j| = |S^D_j|$. Since there is at least one barred dot in this case, we take the sum of this over all j from 0 to $\lfloor (n-1)/2 \rfloor$. Adding this to the result from the case without barred dots, the required formula follows. ■

We remark that the set S_n^H, used in the preceding theorem, is enumerated in (4.1) and (4.2). $|S_n^D|$, also in the preceding theorem, is the number of unsigned involutions of n. In particular, by [15, §3], $|S_0^D| = |S_1^D| = 1$ and, for all $n \geq 2$,

$$|S_n^D| = |S_{n-1}^D| + (n-1)|S_{n-2}^D|. \tag{4.3}$$

The following theorem gives two different enumerations of a set.

Theorem 4.8 *Let $a_n = |B_n^H(\bar{2}\bar{1}, 2\bar{1}, \bar{2}1)|$. Then the following hold.*

(i) For all $n \geq 0$,

$$a_n = |S_n^H| + \sum_{j=1}^{\lceil n/2 \rceil} |S_{n-2j}^H| \cdot |B_{j-1}^D(\bar{2}\bar{1}, 2\bar{1}, \bar{2}1)|,$$

where we use the convention $|S_{-1}^H| = 1$.

(ii) $a_0 = 1$, $a_1 = 2$, and, for all $n \geq 2$,

$$a_n = |S_n^H| + \sum_{j=1}^{\lceil n/2 \rceil} |S_{j-1}^D| \cdot a_{n-2j},$$

where we use the convention $a_{-1} = 1$.

Proof. *(i)* Let $\pi \in B_n^H(\bar{2}\bar{1}, 2\bar{1}, \bar{2}1)$. If π has no barred dots, then none of the restrictions apply, so π can be any permutation in S_n^H. If π has at least one barred dot, then let j be the column of the rightmost barred dot in the left half (including the center column if n is odd). If the jth column's dot were not on the D diagonal, then this dot and its image under D would form a $\bar{2}\bar{1}$, which is forbidden; thus, $\pi(j) = j$ and (by R_{180} invariance) $\pi(n+1-j) = n+1-j$. The columns of π between column j and column $n+1-j$ have only nonbarred dots; further, π avoids $2\bar{1}$ and $\bar{2}1$ if and only if the dots between column j and column $n+1-j$ are also between row j and row $n+1-j$. Thus, the center $(n-2j) \times (n-2j)$ square region of π is any element of S_{n-2j}^H. (In the case where $n - 2j = -1$, there is no center square region, so we set $|S_{-1}^H| = 1$.)

Since $\pi(j) = j$, π avoids $\{\bar{2}\bar{1}, 2\bar{1}, \bar{2}1\}$ if and only if the dots to the left of the jth column are below the jth row. The D' diagonal does not cut through the $(j-1) \times (j-1)$ square region that lies below and to the left of the jth column's dot, so this region must contain an element of B_{j-1}^D. Since π is invariant under R_{180}, choosing this region in the lower-left determines the corresponding region in the upper-right.

We then get the required formula by multiplying the $|S_{n-2j}^H|$ choices for the central square region by the $|B_{j-1}^D|$ choices for the lower-right square region, summing over $1 \leq j \leq \lceil n/2 \rceil$, and adding the $|S_n^H|$ choices from the case where no dots in π are barred.

(ii) Let $\pi \in B_n^H(\bar{2}\bar{1}, 2\bar{1}, \bar{2}1)$. If π has no barred dots, then none of the restrictions apply, so π can be any permutation in S_n^H. If π has at least one barred dot, then let j be the column of the leftmost barred dot. Just as in the proof of *(i)*, $\pi(j) = j$ and $\pi(n+1-j) = n+1-j$. Now, π avoids $\{\bar{2}\bar{1}, 2\bar{1}, \bar{2}1\}$ if and only if the dots between column j and column $n+1-j$ are also between row j and row $n+1-j$. Thus, the center $(n-2j) \times (n-2j)$ square region of π is any element of $B_{n-2j}^H(\bar{2}\bar{1}, 2\bar{1}, \bar{2}1)$, of which there are a_{n-2j} choices. (In the case where $n - 2j = -1$, there is no center square region, so we set $a_{-1} = 1$.)

The part of π to the left of column j has only nonbarred dots. Since $\pi(j) = j$, π avoids $2\bar{1}$ and $\bar{2}1$ if and only if the dots to the left of column j are below row j. The D' diagonal does not cut through the $(j-1) \times (j-1)$ square that lies below and to the left of the jth column's dot, so this region must contain an element of S_{j-1}^D. Since π is invariant under R_{180}, choosing this region in the lower-left determines the corresponding region in the upper-right.

We then get the required formula by multiplying the a_{n-2j} choices for the central square region by the $|S_{j-1}^D|$ choices for the lower-right square region, summing over $1 \leq j \leq \lceil n/2 \rceil$, and adding the $|S_{2k}^H|$ choices from the case where no dots in π are barred. ■

We remark that the preceding theorem used these sets: S_{2k}^H, enumerated in (4.1); $B_n^D(\bar{2}\bar{1}, 2\bar{1}, \bar{2}1)$, enumerated in (6.3); and S_n^D, enumerated in (4.3).

Theorem 4.9 *Let $a_k = |B^H_{2k}(\bar{2}1, \bar{1}2, 1\bar{2}, 2\bar{1})|$. Then $a_0 = 1$, $a_1 = 4$, and, for all $k \geq 2$,*

$$a_k = 2a_{k-1} + 2(k-1)a_{k-2}.$$

Furthermore, for all $k \geq 1$,

$$|B^H_{2k+1}(\bar{2}1, \bar{1}2, 1\bar{2}, 2\bar{1})| = a_k.$$

Proof. For the even case, the base cases are left as an exercise to the reader. For the inductive case, note that if $\pi \in B^H_{2k}(\bar{2}1, \bar{1}2, 1\bar{2}, 2\bar{1})$, everything in π must have the same sign. So choose the location of the leftmost dot, but do not choose the sign (this can be chosen in the base case). Remove the row and column of the leftmost dot and its images under the elements of H, and the recurrence relation follows.

For the odd case, we form a bijection to the evens by adding a middle row and column to each $\pi \in B^H_{2k}(\bar{2}1, \bar{1}2, 1\bar{2}, 2\bar{1})$ and placing a dot of the same sign as everything else in π in the middle square. ■

Theorem 4.10 *For all $n \geq 0$,*

$$|B^H_n(\bar{2}\bar{1}, \bar{2}1, 2\bar{1}, 21)| = 2^{\lceil n/2 \rceil}.$$

Proof. Let $\pi \in B^H_n(\bar{2}\bar{1}, \bar{2}1, 2\bar{1}, 21)$. Everything in π must be increasing, and thus on the D diagonal. We choose the sign of each dot on the left side of π (including the middle if n is odd), which then determines the signs on the right side by the R_{180} invariance. ■

Theorem 4.11 *For all $n \geq 0$, let $a_n = |B^H_n(\bar{2}\bar{1}, \bar{2}1, \bar{1}\bar{2}, 2\bar{1})|$. Then $a_0 = 1$, $a_1 = 2$, $a_2 = 2$, $a_3 = 3$, and, for all $k \geq 2$,*

$$a_{2k} = 2a_{2k-2} + 2(k-1)a_{2k-4}$$

and

$$a_{2k+1} = \frac{1}{2}a_{2k} + a_{2k-1} + (k-1)a_{2k-3}.$$

Proof. The base cases are left as an exercise to the reader.

Let $\pi \in B^H_n(\bar{2}\bar{1}, \bar{2}1, \bar{1}\bar{2}, 2\bar{1})$. In the case of even n, we can have no barred dots because any nonbarred dot has at least 2 images under H, and π avoids $\bar{2}\bar{1}$ and $\bar{1}\bar{2}$. Thus, π is simply an unsigned permutation invariant under H, and so we choose $\pi(1)$ and remove the row and column of all its images under H. If $\pi(1) = 1$ or n, then it has 2 images under H; otherwise it has 4. Thus, in this case, $a_n = 2a_{n-2} + (n-2)a_{n-4}$.

In the case of odd n, π still cannot have more than one bar, but in this case, there is a middle square, which can be of either sign. If the middle square of π is a nonbar, then we can remove the middle row and column and we have reduced the problem to the even case. Thus, there are a_{n-1} permutations of this type. If the middle square of π is a bar, then everything else must be a nonbar, and there cannot be anything in either the upper-left or lower-right quadrants; otherwise there is a $\bar{2}1$ or a $2\bar{1}$. Thus, we fill in an unsigned permutation invariant under D in the lower-left quadrant and fill in its image under D' in the upper-right quadrant.

To get the recurrence relation, note that $a_{2k+1} = a_{2k} + |S^D_k|$, so $|S^D_k| = a_{2k+1} - a_{2k}$. The sequence $|S^D_n|$ satisfies $|S^D_n| = |S^D_{n-1}| + (n-1)|S^D_{n-2}|$ (see [15, §3]). Then

$$a_{2k+1} = a_{2k} + |S^D_{k-1}| + (k-1)|S^D_{k-2}| = a_{2k} + a_{2k-1} - a_{2k-2} + (k-1)(a_{2k-3} - a_{2k-4}).$$

Now we note that by the even case, $-a_{2k-2} - (k-1)a_{2k-4} = -\frac{1}{2}a_{2k}$, which gives the desired result. ■

Theorem 4.12 *For all $n \geq 1$,*

$$|B^H_n(\bar{2}\bar{1}, \bar{2}1, \bar{1}2, 1\bar{2}, 2\bar{1})| = |S^H_n| + 1.$$

Proof. Let $\pi \in B^H_n(\bar{2}\bar{1}, \bar{2}1, \bar{1}2, 1\bar{2}, 2\bar{1})$ If π has a bar, then the entire permutation is increasing barred entries. Otherwise, π is an element of S^H_n. ■

We remark that the set S^H_n, used in the preceding theorem, is enumerated in (4.1) and (4.2).

Theorem 4.13 *For all $n \geq 2$,*

$$B^H_n(\bar{2}\bar{1}, \bar{2}1, \bar{1}\bar{2}, \bar{1}2, 1\bar{2}, 2\bar{1}) = S^H_n.$$

Proof. If $\pi \in B^H_n(\bar{2}\bar{1}, \bar{2}1, \bar{1}\bar{2}, \bar{1}2, 1\bar{2}, 2\bar{1})$, then π has no barred entries, so $\pi \in S^H_n$. If $\pi \in S^H_n$, then of course π has no barred entries, and the result follows. See (4.1) and (4.2) for enumeration. ■

The rest of the results in this section are more straightforward, so we leave their proofs as exercises to the reader. They are listed in Table 3, along with the other results from this section.

n	R	$\lvert B_n^H(R)\rvert$	Recurrence
$2k$ $2k+1$	$\emptyset$		$a_k = 4a_{k-1} + 4(k-1)a_{k-2}$ $b_k = 2a_k$
n	$\{\bar{2}\bar{1}\}$		$a_n = 3a_{n-2} + (n-2)a_{n-4}$
n	$\{\bar{2}\bar{1}, 21\}$	$2^{\lceil n/2 \rceil}$	
n	$\{\bar{2}\bar{1}, 12\}$	$2^{\lceil n/2 \rceil}$	
$2k$ $2k+1$	$\{\bar{2}\bar{1}, \bar{1}\bar{2}\}$		$a_k = 2a_{k-1} + 2(k-1)a_{k-2}$ $b_k = 2a_k$
n	$\{\bar{2}1, 2\bar{1}\}$	Theorem 4.6	
n	$\{\bar{2}\bar{1}, \bar{1}2, 1\bar{2}\}$	$\lvert S_n^H\rvert + \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \lvert S_j^D\rvert$	
n	$\{\bar{2}\bar{1}, 2\bar{1}, \bar{2}1\}$	$\lvert S_n^H\rvert + \sum_{j=1}^{\lceil n/2 \rceil} \lvert S_{n-2j}^H\rvert \cdot \lvert B_{j-1}^D(R)\rvert$	$a_n = \lvert S_n^H\rvert + \sum_{j=1}^{\lceil n/2 \rceil} \lvert S_{j-1}^D\rvert \cdot a_{n-2j}$
$2k$ $2k+1$	$\{\bar{2}1, \bar{1}2, 1\bar{2}, 2\bar{1}\}$		$a_k = 2a_{k-1} + 2(k-1)a_{k-2}$ $b_k = a_k$
n	$\{\bar{2}\bar{1}, \bar{2}1, 2\bar{1}, 21\}$	$2^{\lceil n/2 \rceil}$	
$2k$ $2k+1$	$\{\bar{2}\bar{1}, \bar{2}1, \bar{1}\bar{2}, 2\bar{1}\}$		$a_k = 2a_{k-1} + 2(k-1)a_{k-2}$ $b_k = \frac{1}{2}a_k + b_{k-1} + (k-1)b_{k-2}$
n	$\{\bar{2}\bar{1}, \bar{2}1, \bar{1}2, 1\bar{2}, 2\bar{1}\}$	$\lvert S_n^H\rvert + 1$	
n	$\{\bar{2}\bar{1}, \bar{2}1, \bar{1}\bar{2}, \bar{1}2, 1\bar{2}, 2\bar{1}\}$	$\lvert S_n^H\rvert$	
n	$\{\bar{2}\bar{1}, \bar{1}2, 1\bar{2}, 21\}$	2	
n	$\{\bar{2}\bar{1}, \bar{1}\bar{2}, 12, 21\}$	0	Note: the same is true for any $S \supseteq R$.
n	$\{\bar{2}\bar{1}, \bar{2}1, \bar{1}2, 1\bar{2}, 2\bar{1}, 21\}$	2	
n	$\{\bar{2}\bar{1}, \bar{2}1, \bar{1}2, 1\bar{2}, 12, 2\bar{1}\}$	2	
n	$\{\bar{2}\bar{1}, \bar{2}1, \bar{1}\bar{2}, 12, 2\bar{1}\}$	1	
n	$\{\bar{2}\bar{1}, \bar{2}1, \bar{1}\bar{2}, \bar{1}2, 1\bar{2}, 12, 2\bar{1}\}$	1	
$2k$ $2k+1$	$\{\bar{2}\bar{1}, \bar{1}\bar{2}, 12\}$	1 2	
$2k$ $2k+1$	$\{\bar{2}\bar{1}, \bar{2}1, \bar{1}\bar{2}, 2\bar{1}, 21\}$	1 2	
$2k$ $2k+1$	$\{\bar{2}\bar{1}, \bar{2}1, 12, 2\bar{1}\}$	$k+1$	

Table 3: Enumerations of signed permutations of length at least 4 that are invariant under H and avoid R.

Every $R \subseteq B_2$ that has not been explicitly used as a pattern-avoidance set in this section is equivalent by Lemma 1.1 to one of the sets that has been. Thus, we have now enumerated $B_n^H(R)$ for every $R \subseteq B_2$.

5 Signed permutations invariant under $\overline{D}$

Recall from our introduction that $B_n^{\overline{D}}$ is the set of signed permutations of length n that are invariant under the subgroup $\langle \overline{D} \rangle = \{e, \overline{D}\}$ of $\mathfrak{S}$. Also recall that applying $\overline{D}$ is the same as reflecting across the "/" diagonal (the D diagonal) and applying the bar operation. Equivalently, it is the same as taking the group-theoretic inverse and applying the bar operation. In this section, we enumerate $B_n^{\overline{D}}(R)$ for each $R \subseteq B_2$.

The $\langle \overline{D} \rangle$ orbits of the dots in a diagram must consist of pairs of dots, so a signed permutation of odd length cannot be invariant under $\overline{D}$. That is, $B_{2k+1}^{\overline{D}} = \emptyset$. Now it remains only to enumerate $B_{2k}^{\overline{D}}(R)$ for each $R \subseteq B_2$.

We start by enumerating the permutations invariant under $\overline{D}$ with no pattern avoidance.

Theorem 5.1 *For all $k \geq 0$,*

$$|B_{2k}^{\overline{D}}| = \frac{(2k)!}{k!}.$$

Proof. The $k = 0$ case is easy to verify. Now let $\pi \in B_{2k}^{\overline{D}}$ for $k \geq 1$. There are $2(2k-1)$ choices for the first column's dot: $2k-1$ choices for its position (it cannot go on the D diagonal) and 2 choices for its sign.

This dot determines a dot in the bottom row, by the $\overline{D}$ invariance. Remove the row and column of each of these two dots; what remains is in $B^{\overline{D}}_{2(k-1)}$. Moreover, this process is invertible, up to the choice of the first-column dot: given $\pi' \in B^{\overline{D}}_{2(k-1)}$, we can insert a dot into the first column and insert its image under $\overline{D}$. Thus, for all $k \geq 1$,

$$|B^{\overline{D}}_{2k}| = 2(2k-1)|B^{\overline{D}}_{2(k-1)}|.$$

The required formula follows by induction. ■

Egge [8, Def. 2.1 & Thm. 4.2] constructs a bijection between $B^{\overline{D}}_n$ and S^{90}_{2n}, defined as follows. Given $\pi \in B^{\overline{D}}_n$, copy the $n \times n$ diagram of π into the lower-left quadrant of a $2n \times 2n$ grid. Reflect all of the nonbarred dots across the grid's horizontal center line, moving them into the upper-left quadrant. Then fill in the right half of the $2n \times 2n$ grid with a 180-degree rotation of the left half. The unsigned version of the resulting permutation is in S^{90}_{2n}.

Now we move on to sets with pattern avoidance.

Theorem 5.2 *For all $k \geq 0$,*

$$|B^{\overline{D}}_{2k}(\bar{2}1)| = \frac{(2k)!}{2^k k!}.$$

Proof. If $\pi \in B^{\overline{D}}_{2k}(\bar{2}1)$, then the first dot must be nonbarred, since its image under D is in the bottom row. Furthermore, the first dot cannot be on the diagonal. Thus, choose one of the $2k-1$ other spaces in the first column, and remove the rows and columns that are filled. The rest of the permutation is an element of $B^{\overline{D}}_{2(k-1)}(\bar{2}1)$, and the result follows by induction. ■

Theorem 5.3 *For all $k \geq 0$, let $a_k = |B^{\overline{D}}_{2k}(\bar{1}2, 1\bar{2})|$. Then*

$$a_k = |B_k(\bar{1}2, 1\bar{2})|.$$

It follows that $a_0 = 1$ and, for all $k \geq 1$,

$$a_k = k! + \sum_{j=0}^{k-1} (k-j)! a_j.$$

Proof. Let $\pi \in B^{\overline{D}}_{2k}(\bar{1}2, 1\bar{2})$, and suppose π has a dot in the lower-left quadrant. Then it must also have a dot in the upper-right quadrant, along with its image under $\overline{D}$. But this violates either $\bar{1}2$ or $1\bar{2}$, so everything is in the upper-left and lower-right quadrants.

Since the D diagonal does not pass through either of these quadrants, we simply have to fill in a permutation in the upper-left quadrant that avoids $\bar{1}2, 1\bar{2}$ and then fill in the lower-right quadrant with its image under $\overline{D}$. But this is just an element of $B_k(\bar{1}2, 1\bar{2})$.

Thus, we have a bijection from $B^{\overline{D}}_{2k}(\bar{1}2, 1\bar{2})$ to $B_k(\bar{1}2, 1\bar{2})$: given an element of the former, the upper-left quadrant is an element of the latter. We give a recurrence for $B_k(\bar{1}2, 1\bar{2})$ in Lemma 3.5, and this is the recurrence we need for $B^{\overline{D}}_{2k}(\bar{1}2, 1\bar{2})$. ■

Theorem 5.4 *For all $k \geq 0$,*

$$|B^{\overline{D}}_{2k}(\bar{2}\bar{1}, 21)| = \binom{2k}{k}.$$

Proof. Let $\pi \in B^{\overline{D}}_{2k}(\bar{2}\bar{1}, 21)$. Since π is invariant under $\overline{D}$, π has k nonbarred dots and k barred dots. We are thus choosing k of the $2k$ columns to have nonbarred dots, giving $\binom{2k}{k}$ choices. The nonbarred dots must be in increasing order to avoid 21, and so must be the barred dots to avoid $\bar{2}\bar{1}$. Furthermore, $\bar{D}$ must map the ith nonbarred dot to the ith barred dot for each i. Therefore, given one of the $\binom{2k}{k}$ arrangements of signs, π is uniquely determined. ■

Theorem 5.5 *For all $k \geq 0$,*

$$|B^{\overline{D}}_{2k}(\bar{1}\bar{2}, 12)| = \binom{2k}{k}.$$

Proof. This proof is the same as the proof of Theorem 5.4, but with two modifications. First, the nonbarred dots must be in decreasing order to avoid 12, and so must be the barred dots to avoid $\bar{1}\bar{2}$. Second, $\bar{D}$ must map the ith nonbarred dot to the $(k-i+1)$th barred dot for each i. ■

Theorem 5.6 *For all $k \geq 0$,*

$$|B_{2k}^{\overline{D}}(\bar{2}\bar{1}, \bar{2}1, 21)| = C_k,$$

the kth Catalan number.

Proof. Let $\pi \in B_{2k}^{\overline{D}}(\bar{2}\bar{1}, \bar{2}1, 21)$. Then the barred dots of π are increasing, as are the nonbarred dots. If there is a barred dot above the D diagonal, its image under $\overline{D}$ is a nonbarred dot, so this creates a $\bar{2}1$. Thus, all the barred dots must be below the D diagonal and all the nonbarred dots must be above the D diagonal. Choose k of the $2k$ columns to have nonbarred dots such that every prefix of π contains at least as many nonbarred dots as barred dots. Then if p_j are the columns with nonbarred dots and q_j are the columns with barred dots, we must have $\pi(p_j) = q_j$ and $\pi(q_j) = \overline{p_j}$. Now because the p_j and q_j are all distinct, and because there are k of each of them, there is exactly one dot in each row, so the resulting object is a permutation. Additionally, $\pi(p_j) = q_j$ and $\pi(q_j) = \overline{p_j}$ and these dots are mapped to each other under $\overline{D}$, so π is invariant under $\overline{D}$. Furthermore, since $p_j < q_j$ for each j, the nonbarred dots are all above the diagonal. Thus, we have C_k permutations. ■

Theorem 5.7 *For all $k \geq 0$,*

$$|B_{2k}^{\overline{D}}(\bar{2}1, \bar{1}2, 1\bar{2})| = k!.$$

Proof. If $\pi \in B_{2k}^{\overline{D}}(\bar{2}1, \bar{1}2, 1\bar{2})$, then every nonbarred dot in π must be before and above every barred dot. So, because there are exactly k nonbarred dots, they must all be in the upper-left quadrant. Thus, we simply choose any unsigned permutation of the upper-left quadrant. ■

Theorem 5.8 *For all $k \geq 0$,*

$$|B_{2k}^{\overline{D}}(\bar{1}\bar{2}, \bar{1}2, 1\bar{2}, 12)| = 2^k.$$

Proof. These permutations must be strictly decreasing, but we can choose which of the dots on the left side are barred. ■

The rest of the results in this section are more straightforward, so we leave their proofs as exercises to the reader. They are listed in Table 4, along with the other results from this section.

n	R	$\lvert B_n^{\overline{D}}(R)\rvert$	Recurrence
$2k$	$\emptyset$	$\frac{(2k)!}{k!}$	
$2k$	$\{\bar{2}1\}$	$\frac{(2k)!}{2^k k!}$	
$2k$	$\{\bar{1}2, 1\bar{2}\}$	$\lvert B_k(\bar{1}2, 1\bar{2})\rvert$	$a_k = k! + \sum_{j=0}^{k-1}(k-j)!a_j$
$2k$	$\{\bar{2}\bar{1}, 21\}$	$\binom{2k}{k}$	
$2k$	$\{\bar{1}\bar{2}, 12\}$	$\binom{2k}{k}$	
$2k$	$\{\bar{2}\bar{1}, \bar{2}1, 21\}$	C_k	
$2k$	$\{\bar{2}1, \bar{1}2, 1\bar{2}\}$	$k!$	
$2k$	$\{\bar{1}\bar{2}, \bar{1}2, 1\bar{2}, 12\}$	2^k	
$2k$	$\{2\bar{1}, \bar{2}1\}$	0	Note: the same is true for any $S \supseteq R$.
$2k$	$\{\bar{1}\bar{2}, \bar{2}\bar{1}, 12, 21\}$	0	Note: the same is true for any $S \supseteq R$.
$2k$	$\{\bar{2}\bar{1}, \bar{1}2, 1\bar{2}, 21\}$	2	
$2k$	$\{\bar{2}\bar{1}, \bar{2}1, \bar{1}2, 1\bar{2}, 21\}$	1	
$2k$	$\{\bar{2}1, \bar{1}\bar{2}, 12\}$	1	
$2k$	$\{\bar{2}1, \bar{1}\bar{2}, \bar{1}2, 1\bar{2}, 12\}$	1	

Table 4: Enumerations of signed permutations of length at least 4 that are invariant under $\overline{D}$ and avoid R.

Every $R \subseteq B_2$ that has not been explicitly used as a pattern-avoidance set in this section is equivalent by Lemma 1.1 to one of the sets that has been. Thus, we have now enumerated $B_n^{\overline{D}}(R)$ for every $R \subseteq B_2$.

6 Signed permutations invariant under D and $\overline{D'}$

For this section, let W be the subgroup $\{e, \overline{R_{180}}, D, \overline{D'}\}$ of $\mathfrak{S}$. Recall from our introduction that applying $\overline{R_{180}}$ is the same as taking the reverse–complement and applying the bar operation; D is the same as reflecting across the "/" diagonal (the D diagonal), or the same as taking the group-theoretic inverse; and $\overline{D'} = D \circ \overline{R_{180}} = \overline{R_{180}} \circ D$ is the same as reflecting across the "\" diagonal (the D' diagonal) and applying the bar operation. In this section, we enumerate $B_n^W(R)$ for each $R \subseteq B_2$.

The W orbits of the dots in a diagram must consist of either 2 or 4 dots, depending whether the dot is on the D diagonal. This means that a signed permutation of odd length cannot be invariant under W, the reason being that it cannot be invariant under $\overline{D'}$. That is, $B_{2k+1}^W = \emptyset$. Now it remains only to enumerate $B_{2k}^W(R)$ for each $R \subseteq B_2$.

We start by enumerating the permutations invariant under W with no pattern avoidance. We have two different formulas for this.

Theorem 6.1 *(i)* $|B_0^W| = 1$, $|B_2^W| = 2$, *and, for all* $k \geq 2$,

$$|B_{2k}^W| = 2 \cdot |B_{2(k-1)}^W| + 2(2k-2)|B_{2(k-2)}^W|. \tag{6.1}$$

(ii) For all $k \geq 0$,

$$|B_{2k}^W| = 2^k |S_k^D|. \tag{6.2}$$

Proof. *(i)* The base cases are left as an exercise to the reader. Let $k \geq 1$, and let $\pi \in B_{2k}^W$. Consider the leftmost dot of π. It cannot be in the topmost box because then π would not be invariant under $\overline{D'}$. Remove the leftmost dot and its images under W; if $\pi(1) = \pm 1$, then it has an orbit of size 2; otherwise, the orbit is of size 4. Thus, the recurrence relation holds.

(ii) We could use purely algebraic methods to prove that $2^k|S_k^D|$ follows the recurrence relation in (6.1), since $|S_k^D|$ itself follows a similar recurrence relation [see (4.3)]. However, we instead give a combinatorial proof of (6.2).

For all $n \geq 0$, let J_n be the set of signed permutations whose unsigned versions are invariant under D. We can construct elements of J_n by giving signs to the entries of elements of S_n^D. There are 2^n choices of signs for each element of S_n^D, so $|J_n| = 2^n|S_n^D|$.

We construct a bijection from J_n to B_{2n}^W as follows. Given $\pi \in J_n$, copy the $n \times n$ diagram of π into the lower-left quadrant of a $2n \times 2n$ grid. If two dots in the same D orbit have opposite signs, then reflect the pair across the grid's horizontal center line, putting it into the upper-left quadrant. Then fill in the right half of the $2n \times 2n$ grid with a 180-degree rotation of the left half. Now flip the sign of all dots in the right half. It is easy to check that this map sends π to a signed permutation in B_{2n}^W. Since the process is invertible, it is a bijection. Therefore, $|B_{2k}^W| = |J_k| = 2^k|S_k^D|$. ■

Now we move on to sets with pattern avoidance.

Theorem 6.2 $|B_0^W(\bar{1}2)| = |B_2^W(\bar{1}2)| = 1$ *and, for all* $k \geq 2$,

$$|B_{2k}^W(\bar{1}2)| = |B_{2(k-1)}^W(\bar{1}2)| + 2(k-1)|B_{2(k-2)}^W(\bar{1}2)|.$$

Proof. The proof of Theorem 6.1 *(i)* also works here, with the exception that now we have no choice of sign when we insert a dot. This is because inserting a barred dot anywhere in the far left column would result in a $\bar{1}2$, with the bar's image under $\overline{D'}$. So the present recurrence relation's coefficients are half what they are in the unrestricted case. ■

Theorem 6.3 *For all* $k \geq 0$,

$$|B_{2k}^W(\bar{1}\bar{2}, 12)| = 2^k.$$

Proof. Let $\pi \in B_{2k}^W(\bar{1}\bar{2}, 12)$. For each entry in the first half of π, choose whether it is barred. There are 2^k ways to choose this, and this choice determines the placement of barred entries in the right half. We will now prove inductively that, given the choice of bars, π is uniquely determined.

The base cases are easy to check. Now, consider $\pi \in B_{2k}^W(\bar{1}\bar{2}, 12)$ for $k \geq 2$. Choose the bars' placement in π, as described above; without loss of generality, we choose the first entry to be nonbarred. To avoid

$\bar{1}\bar{2}$ and 12, we must put $2k$ as either the first nonbarred entry or the first barred entry. By $\overline{D'}$ invariance, $\pi(1) \neq 2k$, and so $2k$ must be the first barred entry. We already chose the colors, so we know that the first bar is on the jth entry of π, where $2 \leq j \leq k+1$. By W invariance, we now have $\pi(j) = \overline{2k}$; $\pi(2k) = \bar{j}$; $\pi(2k-j+1) = 1$; and $\pi(1) = 2k-j+1$.

Remove these four entries from π; let ρ be the permutation that remains. Note that $\rho \in B^W_{2(k-2)}(\bar{1}\bar{2}, 12)$. By the inductive hypothesis, ρ is uniquely determined by a choice of bar placement, so let ρ inherit the choice that we made for π. Now re-insert the 4 entries to recover π.

Now π has been uniquely determined, and it is easy to check that π is still invariant under W. We know by the inductive hypothesis that ρ has the required pattern restriction, but perhaps an entry of ρ forms a forbidden pattern with the $\overline{2k}$ in position j of π. Suppose this is the case. Then there must be a barred entry before position j, a contradiction. So position j cannot be in a forbidden pattern. Furthermore, any possible forbidden pattern that includes one of the other three initial entries can be mapped by W to a forbidden pattern that includes position j, which would also be a contradiction. Therefore, π really does obey the necessary restrictions. ■

Theorem 6.4 *For all $k \geq 0$,*

$$|B^W_{2k}(\bar{2}\bar{1}, 21)| = 2^k.$$

Proof. Let $\pi \in B^W_{2k}(\bar{2}\bar{1}, 21)$. If π has a dot not on the D diagonal, then this dot, along with its image under D, creates a $\bar{2}\bar{1}$ or a 21. Thus, everything is on the diagonal of π, so we choose the colors of the dots on the left half. The right half is determined by $\overline{D'}$ invariance. ■

Theorem 6.5 *Let $a_k = |B^W_{2k}(\bar{2}1, 2\bar{1})|$. Then, for all $k \geq 0$,*

$$a_k = |B^D_k(\bar{2}1, 2\bar{1})|.$$

Furthermore, a_k is given by the recurrence relation $a_0 = 1$, $a_1 = 2$, and, for all $k \geq 2$,

$$a_k = 2a_{k-1} + ka_{k-2} - \sum_{j=1}^{k-3} j \cdot |S^D_{k-j-3}| a_j. \tag{6.3}$$

Proof. Dukes and Mansour prove that the recurrence relation (6.3) enumerates $B^D_k(\bar{2}1, 2\bar{1})$ [5, Prop. 3.8]. Hence, we have only to prove that $|B^W_{2k}(\bar{2}1, 2\bar{1})| = |B^D_k(\bar{2}1, 2\bar{1})|$.

If $\pi \in B^W_{2k}(\bar{2}1, 2\bar{1})$ and there is a dot in the upper-left quadrant of π, then the image of this dot under $\overline{D'}$ is also in the upper-left quadrant, and its image under D is in the lower-right quadrant. These two dots create a $\bar{2}1$ or a $2\bar{1}$, so everything is in the lower-left and upper-right quadrants. Thus, we fill in a signed involution in the lower-left quadrant avoiding $\bar{2}1$ and $2\bar{1}$, and fill in the upper-right quadrant using $\overline{D'}$. ■

We remark that the preceding theorem used the set S^D_n, the number of unsigned involutions, as enumerated in (4.3).

Theorem 6.6 *For all $k \geq 0$,*

$$|B^W_{2k}(\bar{1}\bar{2}, \bar{1}2, 12)| = \binom{k}{\lfloor k/2 \rfloor}.$$

Proof. For this proof, let $\mathcal{A}_k = B^W_{2k}(\bar{1}\bar{2}, \bar{1}2, 12)$, let $H = \{e, R_{180}, D, D'\} \subseteq D_4$, and let

$$\mathcal{D}_k = \{\pi \in S^H_{2k}(123) : \text{the diagram of } \pi \text{ has no dot on the } D' \text{ diagonal}\}.$$

The enumeration of $\mathcal{A}_k$ has three steps. *I.* We construct a bijection between $\mathcal{A}_k$ and $\mathcal{D}_k$. *II.* We find a recursive formula that enumerates $\mathcal{D}_k$. *III.* We use ordinary generating functions to show that the recursive formula yields the zigzag central binomial coefficients, $\binom{k}{\lfloor k/2 \rfloor}$.

I. Given a signed permutation $\pi \in \mathcal{A}_k$, let $|\pi|$ be the unsigned version of π. Since π is invariant under W, the unsigned $|\pi|$ is invariant under H. If $|\pi|$ were to contain 123, then by the pigeonhole principle two of the dots in the 123 would have the same sign in π, which would make π fail to avoid 12 or $\bar{1}\bar{2}$; thus, $|\pi|$ avoids 123. Because of the $\overline{D'}$ invariance, no dot in π can be on the D' diagonal, so the same is true for $|\pi|$. Thus, $\pi \mapsto |\pi|$ is a map from $\mathcal{A}_k$ to $\mathcal{D}_k$.

If $\pi \in \mathcal{A}$ had a barred dot somewhere below its D' diagonal, then this dot would form a $\bar{1}2$ with its image under $\overline{D'}$, and this is forbidden; thus, every dot below the D' diagonal is nonbarred, and every dot above the D' diagonal is barred. So, given the positions of the dots of π, the signs are uniquely determined. This means we can invert the map defined above: given $|\pi| \in \mathcal{D}_k$, π is the result of putting bars on the dots of $|\pi|$ that are above the D' diagonal. Therefore, the map is a bijection between $\mathcal{A}_k$ and $\mathcal{D}_k$.

So, to count $\mathcal{A}_k$, we need only count $\mathcal{D}_k$. (For the rest of the proof, we will denote an element of $\mathcal{D}_k$ as π rather than $|\pi|$.)

II. For $1 \leq j \leq k$, let $\mathcal{D}_k^j$ be the set of $\pi \in S_{2k}^H(123)$ such that column j of π is the rightmost left-half column that has its dot on the D' diagonal. That is, if $\pi \in \mathcal{D}_k^j$, then column j of π has its dot on the D' diagonal, but no column of π between j and $2k - j$ has its dot on the D' diagonal.

Let $\pi \in \mathcal{D}_k^j$. If π had a dot below and to the left of the jth-column dot, then these two dots would form a 123 with the former's image under D', and this is forbidden; thus, every dot to the left of the jth-column dot is above it. This requires that the upper-left $(j-1) \times (j-1)$ grid contain an unsigned permutation invariant under D' that avoids 123. The reverse map is a natural bijection between the set of such permutations and the set $S_{j-1}^D(321)$. Simion and Schmidt [17, Prop. 5] prove that $|S_n^D(321)| = \binom{n}{\lfloor n/2 \rfloor}$; thus, there are $\binom{j-1}{\lfloor (j-1)/2 \rfloor}$ possibilities for the upper-left $(j-1) \times (j-1)$ grid.

The remaining dots of π must be in the $2(k-j) \times 2(k-j)$ grid in the center. Column j had the last dot that was on the D' diagonal, so none of these dots are on the D' diagonal. This means that we have an unsigned permutation in $\mathcal{D}_{k-j}$, the subset of $S_{2(k-j)}^H(123)$ with no dot on the D' diagonal. Now we can write a formula for $|\,\mathcal{D}_k^j\,|$:

$$|\,\mathcal{D}_k^j\,| = \binom{j-1}{\lfloor (j-1)/2 \rfloor} |\,\mathcal{D}_{k-j}\,|. \tag{6.4}$$

Now, to enumerate $\mathcal{D}_k$, we can count the elements of $S_{2k}^H(123)$ and subtract all of the elements that have a dot on the D' diagonal. Using (6.4) and the fact that $|S_{2k}^H(123)| = 2^k$ [8, Thm. 3.8],

$$\begin{aligned} |\,\mathcal{D}_k\,| &= |S_{2k}^H(123)| - \sum_{j=1}^{k} |\,\mathcal{D}_k^j\,| \\ &= 2^k - \sum_{j=0}^{k-1} \binom{k-j-1}{\lfloor (k-j-1)/2 \rfloor} |\,\mathcal{D}_j\,|. \end{aligned}$$

Therefore, letting $a_k = |\,\mathcal{D}_k\,|$, we have $a_0 = 1$ and, for all $k \geq 1$,

$$a_k = 2^k - \sum_{j=0}^{k-1} \binom{k-j-1}{\lfloor (k-j-1)/2 \rfloor} a_j. \tag{6.5}$$

The unique sequence that follows this recurrence relation is the sequence that enumerates $B_{2k}^W(\bar{1}\bar{2}, \bar{1}2, 12)$.

III. We now prove that $a_k = \binom{k}{\lfloor k/2 \rfloor}$ using ordinary generating functions. Let $A(x) = \sum_{k=0}^{\infty} a_k x^k$, the generating function of a_k. We multiply recurrence relation (6.5) by x^k and sum over all $k \geq 1$:

$$\begin{aligned} A(x) &= 1 + \sum_{k=1}^{\infty} 2^k x^k - \sum_{k=1}^{\infty} \left[\sum_{j=0}^{k-1} \binom{k-j-1}{\lfloor (k-j-1)/2 \rfloor} \cdot a_j \right] x^k \\ &= \sum_{k=0}^{\infty} 2^k x^k - x \left[\sum_{k=0}^{\infty} \binom{k}{\lfloor k/2 \rfloor} x^k \right] A(x). \end{aligned}$$

It is easy to check that the zigzag central binomial coefficients have generating function $\frac{1+2x-\sqrt{1-4x^2}}{2x\sqrt{1-4x^2}}$. Substituting this, as well as substituting $A(x)$ and $\sum_{k \geq 0} 2^k x^k = 1/(1-2x)$, yields the following:

$$A(x) = \frac{1}{1-2x} - \frac{1+2x-\sqrt{1-4x^2}}{2\sqrt{1-4x^2}} A(x).$$

Finally, solving for $A(x)$ and simplifying yields:

$$A(x) = \frac{1+2x-\sqrt{1-4x^2}}{2x\sqrt{1-4x^2}},$$

the same generating function as the zigzag central binomial coefficients. Therefore, $a_k = \binom{k}{\lfloor k/2 \rfloor}$. ■

Theorem 6.7 *Let $a_k = |B_{2k}^W(\bar{2}1, \bar{1}2, 2\bar{1})|$. Then $a_0 = a_1 = 1$ and, for all $k \geq 2$,*

$$a_k = a_{k-1} + (k-1)a_{k-2}.$$

Proof. The base cases are left as an exercise to the reader. If $k \geq 2$, and $\pi \in B_{2k}^W(\bar{2}1, \bar{1}2, 2\bar{1})$, the nonbarred dots of π must all be before and lower than the barred dots; since exactly half of the dots are barred, they must be in the lower-left quadrant. Therefore, we simply fill in the lower-left quadrant with an element of S_k^D and then use $\overline{D'}$ to fill in the upper-right quadrant.

Thus, $a_k = |S_k^D|$, the number of unsigned involutions of k. The recurrence relation is the same as that given for $|S_k^D|$ in equation (4.3). ■

Theorem 6.8 *For all $k \geq 0$,*

$$|B_{2k}^W(\bar{2}\bar{1}, \bar{2}1, 2\bar{1}, 21)| = 2^k.$$

Proof. Let $\pi \in B_{2k}^W(\bar{2}\bar{1}, \bar{2}1, 2\bar{1}, 21)$. Then everything in π is increasing, (and thus on the D diagonal), so we choose the signs of the dots in the left half; the right half is now determined by $\overline{D'}$ invariance. ■

The rest of the results in this section are more straightforward, so we leave their proofs as exercises to the reader. They are listed in Table 5, along with the other results from this section.

n	R	$\lvert B_n^W(R)\rvert$	Recurrence
$2k$	$\emptyset$	$2^k\lvert S_k^D\rvert$	$a_k = 2a_{k-1} + 2(2k-2)a_{k-2}$
$2k$	$\{\bar{1}2\}$		$a_k = a_{k-1} + 2(k-1)a_{k-2}$
$2k$	$\{\bar{1}\bar{2}, 12\}$	2^k	
$2k$	$\{\bar{2}\bar{1}, 21\}$	2^k	
$2k$	$\{\bar{2}1, 2\bar{1}\}$	$\lvert B_k^D(\bar{2}1, 2\bar{1})\rvert$	$a_k = 2a_{k-1} + ka_{k-2} - \sum_{j=1}^{k-3} j \cdot \lvert S_{k-j-3}^D\rvert a_j$
$2k$	$\{\bar{1}\bar{2}, \bar{1}2, 12\}$	$\binom{k}{\lfloor k/2\rfloor}$	
$2k$	$\{\bar{2}1, \bar{1}2, 2\bar{1}\}$		$a_k = a_{k-1} + (k-1)a_{k-2}$
$2k$	$\{\bar{2}\bar{1}, \bar{2}1, 2\bar{1}, 21\}$	2^k	
$2k$	$\{\bar{1}2, 1\bar{2}\}$	0	Note: the same is true for any $S \supseteq R$.
$2k$	$\{\bar{1}\bar{2}, \bar{2}\bar{1}, 12, 21\}$	0	Note: the same is true for any $S \supseteq R$.
$2k$	$\{\bar{2}1, \bar{1}2, 12, 2\bar{1}\}$	2	
$2k$	$\{\bar{2}1, \bar{1}\bar{2}, \bar{1}2, 12, 2\bar{1}\}$	1	
$2k$	$\{\bar{2}\bar{1}, \bar{1}2, 21\}$	1	
$2k$	$\{\bar{2}\bar{1}, \bar{2}1, \bar{1}2, 2\bar{1}, 21\}$	1	

Table 5: Enumerations of signed permutations of length at least 4 that are invariant under W and avoid R.

Every $R \subseteq B_2$ that has not been explicitly used as a pattern-avoidance set in this section is equivalent by Lemma 1.1 to one of the sets that has been. Thus, we have now enumerated $B_n^W(R)$ for every $R \subseteq B_2$.

7 Signed permutations invariant under $\overline{D}$ and $\overline{D'}$

For this section, let $\overline{H}$ be the subgroup $\{e, R_{180}, \overline{D}, \overline{D'}\}$ of $\mathfrak{S}$. Recall from our introduction that applying $\overline{D}$ is the same as reflecting across the "/" diagonal (the D diagonal) and applying the bar operation. Equivalently, it is the same as taking the group-theoretic inverse and applying the bar operation. Also recall that R_{180} is the same as taking the reverse–complement, and $\overline{D'} = \overline{D} \circ R_{180} = R_{180} \circ \overline{D}$ is the same as reflecting across the "\" diagonal (the D' diagonal) and applying the bar operation. In this section, we enumerate $B_n^{\overline{H}}(R)$ for each $R \subseteq B_2$.

For $\pi \in B_n^{\overline{H}}$, no dot in π can be mapped to itself by $\overline{D}$ nor by $\overline{D'}$, since these are sign-flipping operations. This means that π cannot have dots on either of its diagonals; in particular, there cannot be a central dot. Then no dot in π can be mapped to itself by R_{180}. Thus, each of the $\langle\overline{H}\rangle$ orbits of dots in π must contain exactly 4 dots. So $B_n^{\overline{H}} = \emptyset$ for any $n \not\equiv 0 \pmod 4$. Thus, it remains only to enumerate $B_n^{\overline{H}}(R)$ for $n \equiv 0 \pmod 4$.

We start by enumerating the permutations invariant under $\overline{H}$ with no pattern avoidance.

Theorem 7.1 *For all $k \geq 0$,*

$$B_{4k}^{\overline{H}} = \frac{2^k(2k)!}{k!}.$$

Proof. The case with $k = 0$ is easy to verify. Now, given a signed permutation in $B_{4k}^{\overline{H}}$ for $k \geq 1$, look at where the dot in the first column is. No dot can be on either of the two diagonals, so there are $4k - 2 = 2(2k-1)$ choices for its position and 2 choices for its sign. So there are $4(2k-1)$ choices for the first column. This dot's orbit under $\overline{H}$ is of size 4, so choosing one dot determines 4 dots in all. If we remove the row and column of each of these 4 dots, then what remains is a signed permutation in $B_{4(k-1)}^{\overline{H}}$. Moreover, this process is invertible up to choice of the leftmost column's dot: given $\pi' \in B_{4(k-1)}^{\overline{H}}$, we can insert a dot somewhere on the far left and then insert the three other dots that are determined by the $\overline{H}$ invariance condition. Thus,

$$|B_{4k}^{\overline{H}}| = 4(2k-1)|B_{4(k-1)}^{\overline{H}}|$$

for all $k \geq 1$. The required formula follows by induction. ■

Now we move on to sets with pattern avoidance.

Theorem 7.2 *For all $k \geq 0$, we have*

$$|B_{4k}^{\overline{H}}(\bar{2}\bar{1}, 21)| = \binom{2k}{k}.$$

Proof. Let $\pi \in B_{4k}^{\overline{H}}(\bar{2}\bar{1}, 21)$, and suppose π has a dot in the upper-left quadrant. Then the image under R_{180} of this dot is in the lower-right quadrant of π, and this creates a $\bar{2}\bar{1}$ or a 21. Thus, everything is in the lower-left and upper-right quadrants, so we choose a permutation for the lower left quadrant avoiding $\bar{2}\bar{1}$ and 21 and invariant under $\overline{D}$, and then fill in its image under $\overline{D'}$ in the upper-right quadrant. There are $\binom{2k}{k}$ ways to do this, by Theorem 5.4. ■

The rest of the results in this section are more straightforward, so we leave their proofs as exercises to the reader. They are listed in Table 6, along with the other results from this section.

n	R	$\lvert B_n^{\overline{H}}(R)\rvert$
$4k$	$\emptyset$	$\frac{2^k(2k)!}{k!}$
$4k$	$\{\bar{2}\bar{1}, 21\}$	$\binom{2k}{k}$
$4k$	$\{\bar{2}1, 2\bar{1}\}$	0
$4k$	$\{\bar{1}2, 1\bar{2}\}$	0
$4k$	$\{\bar{1}\bar{2}, \bar{2}\bar{1}, 12, 21\}$	0

Table 6: Enumerations of signed permutations of length at least 4 that are invariant under $\overline{H}$ and avoid R.

Every $R \subseteq B_2$ that has not been explicitly used as a pattern-avoidance set in this section is equivalent by Lemma 1.1 to one of the sets that has been. Thus, we have now enumerated $B_n^{\overline{H}}(R)$ for every $R \subseteq B_2$.

8 Signed permutations invariant under R_{90} and signed permutations invariant under $\overline{R_{90}}$

Let B_n^{90} be the set of signed permutations of length n that are invariant under the subgroup $\langle R_{90}\rangle = \{e, R_{90}, R_{180}, R_{270}\}$ of $\mathfrak{S}$. Similarly, let $B_n^{\overline{90}}$ be the set invariant under $\langle\overline{R_{90}}\rangle = \{e, \overline{R_{90}}, R_{180}, \overline{R_{270}}\}$ of $\mathfrak{S}$. In this section, we enumerate $B_n^{90}(R)$ and $B_n^{\overline{90}}(R)$ for each $R \subseteq B_2$.

For $\pi \in B_n^{90}$, each of the $\langle R_{90} \rangle$ orbits of dots in π must contain exactly 4 dots, with the possible exception of a central dot which is in an orbit by itself. It follows that $B_n^{90} = \emptyset$ for any $n \equiv 2$ or 3 (mod 4). For $\pi \in B_n^{\overline{90}}$, the same is true, except that there cannot be a dot in the center in an orbit by itself. So $B_n^{\overline{90}} = \emptyset$ for any $n \not\equiv 0$ (mod 4). Thus, it remains only to enumerate $B_n^{90}(R)$ for $n \equiv 0$ or 1 (mod 4) and to enumerate $B_n^{\overline{90}}(R)$ for $n \equiv 0$ (mod 4).

Theorem 8.1 *For all $k \geq 0$,*

$$|B_{4k}^{90}| = |B_{4k}^{\overline{90}}| = \frac{2^k(2k)!}{k!}$$

and

$$|B_{4k+1}^{90}| = 2 \cdot |B_{4k}^{90}| = \frac{2^{k+1}(2k)!}{k!}.$$

Proof. It is easy to verify the cases where $k = 0$. Now, given $\pi \in B_{4k}^{90}$ with $k \geq 1$, choose a dot to go in the first column. No corner box can have a dot in it, so there are $2(4k-2)$ choices for the first column: $4k-2$ position choices and 2 signs. By the R_{90} invariance of π, this dot determines one dot in the last column, one in the bottommost row, and one in the topmost row. If we remove the row and column of each of these four dots, then what remains is in $B_{4(k-1)}^{90}$. Moreover, this process is invertible, up to choice of the leftmost column's dot: given $\pi' \in B_{4(k-1)}^{90}$, we can insert a dot somewhere on the far left and then insert the three other dots that are determined by the R_{90} invariance condition. Thus, for all $k \geq 1$,

$$|B_{4k}^{90}| = 2(4k-2)|B_{4(k-1)}^{90}|.$$

The required formula follows by induction.

To prove that $|B_{4k}^{90}| = |B_{4k}^{\overline{90}}|$, note that the following is a bijection: given $\pi \in |B_{4k}^{90}|$, flip the sign on all dots in the upper-left and lower-right quadrants.

Let $\pi \in B_{4k+1}^{90}$ for any k. Because of the R_{180} invariance, there must be a dot in the central box of the diagram, with 2 choices for its sign. Remove the row and column of this dot; what remains is in B_{4k}^{90}. Therefore, $|B_{4k+1}^{90}| = 2 \cdot |B_{4k}^{90}|$. ■

Now we move on to sets with pattern avoidance.

Theorem 8.2 *For all $k \geq 1$, we have*

$$|B_{4k}^{90}(\bar{2}\bar{1}, \bar{2}1, \bar{1}\bar{2}, \bar{1}2, 1\bar{2}, 2\bar{1})| = |B_{4k+1}^{90}(\bar{2}\bar{1}, \bar{2}1, \bar{1}\bar{2}, \bar{1}2, 1\bar{2}, 2\bar{1})| = \frac{(2k)!}{k!}, \tag{8.1}$$

$$|B_{4k}^{90}(\bar{2}\bar{1}, \bar{1}\bar{2})| = \frac{(2k)!}{k!} \text{ and } |B_{4k+1}^{90}(\bar{2}\bar{1}, \bar{1}\bar{2})| = 2 \cdot \frac{(2k)!}{k!}, \tag{8.2}$$

and

$$|B_{4k}^{90}(\bar{2}1, \bar{1}2, 1\bar{2}, 2\bar{1})| = |B_{4k+1}^{90}(\bar{2}1, \bar{1}2, 1\bar{2}, 2\bar{1})| = 2 \cdot \frac{(2k)!}{k!}. \tag{8.3}$$

Proof. For (8.1), if $\pi \in B_n^{90}(\bar{2}\bar{1}, \bar{2}1, \bar{1}\bar{2}, \bar{1}2, 1\bar{2}, 2\bar{1})$, then every dot in π is nonbarred. Thus, each element of $|B_{4k}^{90}(\bar{2}\bar{1}, \bar{2}1, \bar{1}\bar{2}, \bar{1}2, 1\bar{2}, 2\bar{1})|$ and $|B_{4k}^{90}(\bar{2}\bar{1}, \bar{2}1, \bar{1}\bar{2}, \bar{1}2, 1\bar{2}, 2\bar{1})|$ corresponds to 2^k elements of $|B_{4k}^{90}|$ and $|B_{4k+1}^{90}|$, respectively, by changing every dot to a nonbarred dot.

The proof of (8.2) is the same as that of equation (8.1) except that in the $4k+1$ case we can choose the sign of the center dot. The proof of (8.3) is the same as that of (8.1) except that either all the dots are barred or all of the dots are nonbarred. ■

Here, we look at the only nonempty set of patterns to avoid that is nontrivial in conjunction with $\overline{R_{90}}$.

Theorem 8.3 *For all $k \geq 1$,*

$$|B_{4k}^{\overline{90}}(\bar{2}\bar{1}, 12)| = \binom{2k}{k}.$$

Proof. Let $\pi \in B_{4k}^{\overline{90}}(\bar{2}\bar{1}, 12)$. If there were a barred dot in the upper-left or lower-right quadrant of π, then it would form a $\bar{2}\bar{1}$ with its image under R_{180}, which is forbidden; thus, all of the barred dots of π are in the lower-left quadrant and the upper-right quadrant. Similarly, the nonbarred dots are in the other two quadrants. Choose k of the $2k$ columns in the left half of π to have barred dots and the others to have nonbarred dots; this determines the right half's columns by R_{180}. Furthermore, because π is invariant under $\overline{R_{90}}$, the indices of columns containing barred dots are the indices of the rows containing nonbarred dots, so place the barred dots in increasing order, which determines the nonbarred dots in decreasing order. ■

The rest of the results in this section are more straightforward, so we leave their proofs as exercises to the reader. They are listed in Tables 7 and 8, along with the other results from this section.

n	R	$\lvert B_n^{90}(R)\rvert$
$4k$ $4k+1$	$\emptyset$	$\frac{2^k(2k)!}{k!}$ $\frac{2^{k+1}(2k)!}{k!}$
$4k$ $4k+1$	$\{\bar{2}\bar{1}, \bar{2}1, \bar{1}\bar{2}, \bar{1}2, 1\bar{2}, 2\bar{1}\}$	$\frac{(2k)!}{k!}$
$4k$ $4k+1$	$\{\bar{2}\bar{1}, \bar{1}\bar{2}\}$	$\frac{(2k)!}{k!}$ $2 \cdot \frac{(2k)!}{k!}$
$4k$ $4k+1$	$\{\bar{2}1, \bar{1}2, 1\bar{2}, 2\bar{1}\}$	$2 \cdot \frac{(2k)!}{k!}$
n	$\{\bar{2}\bar{1}, \bar{1}\bar{2}, 12, 21\}$	0

Table 7: Enumerations of signed permutations of length at least 4 that are invariant under R_{90} and avoid R.

n	R	$\lvert B_n^{\overline{90}}(R)\rvert$
$4k$	$\emptyset$	$\frac{2^k(2k)!}{k!}$
$4k$	$\{\bar{2}\bar{1}, 12\}$	$\binom{2k}{k}$
$4k$	$\{\bar{2}1, 2\bar{1}\}$	0
$4k$	$\{\bar{1}\bar{2}, \bar{2}\bar{1}, 12, 21\}$	0

Table 8: Enumerations of signed permutations of length at least 4 that are invariant under $\overline{R_{90}}$ and avoid R.

Every $R \subseteq B_2$ that has not been explicitly used as a pattern-avoidance set in this section is equivalent by Lemma 1.1 to one of the sets that has been. Thus, we have now enumerated $B_n^{90}(R)$ and $B_n^{\overline{90}}(R)$ for every $R \subseteq B_2$.

9 Avoiding longer patterns

Now that we have completed the task of enumerating $B_n^H(R)$ for all $H \leq \mathfrak{S}$ and all $R \subseteq B_2$, we present two sporadic results that are applicable in certain cases where R is not a subset of B_2.

Theorem 9.1 *Let S be a set of permutations whose entries are all nonbarred, let T be a set of permutations whose entries are all barred, and let $H \leq D_4$ be one of the following subgroups:*

$$\{e\}, \quad \{e, D\}, \quad \{e, R_{180}\}, \quad \{e, R_{180}, D, D'\}, \quad \{e, R_{90}, R_{180}, R_{270}\}.$$

Then, for all $n \geq 0$,

$$|B_n^H(S \cup T)| = \sum_{j=0}^{n} A(n, j, H)|S_j^H(S)||S_{n-j}^H(T)|,$$

where

$$A(n,j,H) = \begin{cases} \binom{n}{j}^2 & \text{if } H = \{e\}, \\ \binom{n}{j} & \text{if } H = \{e, D\}, \\ \binom{\lfloor n/2 \rfloor}{\lfloor j/2 \rfloor}^2 \chi(j \text{ is even or } n-j \text{ is even }) & \text{if } H = \{e, R_{180}\}, \\ \binom{\lfloor n/2 \rfloor}{\lfloor j/2 \rfloor} \chi(j \text{ is even or } n-j \text{ is even }) & \text{if } H = \{e, R_{180}, D, D'\} \text{ or } \{e, R_{90}, R_{180}, R_{270}\}. \end{cases}$$

Proof. In every case, we can construct every permutation in $B_n^H(S \cup T)$ uniquely in the following way. First, choose a permutation π of size j with no barred entries that avoids S and a permutation ρ of size $n-j$ with no nonbarred entries that avoids T. We intertwine these permutations by choosing the rows and columns for π and leaving the remaining rows and columns for ρ. Since no entry in any pattern in S is barred and π avoids S, the resulting permutation also avoids S. Similarly, the resulting permutation also avoids T. Thus, we must determine how many ways we can intertwine π and ρ to create a permutation σ fixed by H.

If $H = \{e\}$, then we can choose the rows and columns for π arbitrarily. If $H = \{e, D\}$, then we can choose the rows for π, and this determines the columns.

If $H = \{e, R_{180}\}$, then if j and $n-j$ are both odd, they both have an entry in their middle square, which must occupy the middle square of σ in order for σ to be fixed by H. However, σ does not have a middle square (and certainly not two), so this is impossible. If at most one of j and $n-j$ is odd, then remove the middle square (if it exists) from either π or ρ. Then choose the $\frac{j}{2}$ rows in the lower half and the $\frac{j}{2}$ columns in the left half for π. This determines the columns on the right half and the rows in the upper half. Finally, if there was a middle square for either π or ρ, insert it into the middle of σ.

If $H = \{e, R_{180}, D, D'\}$ or $\{e, R_{90}, R_{180}, R_{270}\}$, then we can make a similar argument except that we choose only the rows for π. ■

Signed permutations can be thought of as two-colored permutations, so the natural generalization is to enumerate sets of r-colored permutations for arbitrary r. Some work has been done on this already. For example, Mansour [11] enumerates the r-colored permutations that avoid ρ for any colored pattern ρ of length 2. Bagno, Garber, and Mansour [1] study the joint distribution of fixed points and a certain type of exceedance on r-colored involutions, and then Mansour and Sun [13] generalize this to r-colored permutations σ for which σ^m is the identity. Egge [7] enumerates r-colored permutations and involutions with some particular pattern avoidances, generalizing some earlier results about Chebyshev polynomials.

Here, we extend the results of Mansour and enumerate $rS_n(\rho)$, where rS_n is the set of $r-$colored permutations of n and $|\rho| = k(k-1)\dots321$ for some k. First, we need a definition.

Definition 9.2 *Let $T(r,k)$ be the generating tree with labels $(n, a_0 = 0, a_1, \dots, a_k)$, and root node $(0, 0, \infty, \dots, \infty)$. The rules for the children of a node $(n, a_1, \dots, a_k)$ are as follows.*

- *For each $j < k$, the node with label $(n, a_0, a_1, \dots, a_k)$ has $(r-1)(a_j - a_{j-1})$ children with label $(n+1, a_0, a_1, \dots, a_j, a_{j+1}+1, \dots, a_k+1)$*
- *For each $j < k-1$, $m \le a_j - a_{j-1}$, the node with label $(n, a_0, a_1, \dots, a_k)$ has one child with label $(n+1, a_0, a_1, \dots, a_j, a_j+m, a_{j+2}+1, \dots, a_k+1)$.*

Theorem 9.3 *If $\rho \in rS_k$ and $|\rho| = k(k-1)\dots21$, then the generating tree for $rS_n(\rho)$ is isomorphic to $T(r,k)$.*

Proof. Let T be the generating tree for $rS_n(\rho)$. For each $\pi \in rS_n(\rho)$, we create the label $(n, a_0, a_1, \dots, a_k)$ in the following way. Let n be the length of π and let $a_0 = 0$. Further, for $1 \le j \le k$, let a_j denote the smallest number so that the length-a_j prefix of π contains the length-j prefix of ρ. If no such number exists, let $a_j = \infty$. By this definition, the root nodes of T and $T(r,k)$ are the same. Now let $v = (n, a_0, a_1, \dots, a_k)$ be a node in T. Then, to get the leaves of v we can add a dot of some color to the bottom of π in column c. Since the a_j are distinct, there exists a j so that $a_j < c \le a_{j+1}$. We have two cases. If the added dot is the same color as the $(j+1)$th entry of ρ, then the label of the resulting child is $(n+1, a_0, a_1, \dots, a_j, c, a_{j+2}+1, \dots, a_k+1)$, and there is one way this can happen for each value of c. We prune this child if $j+1 = k$ because that corresponds to a permutation containing ρ. If the added dot is a different color from the $(j+1)$th entry of ρ, then the label of the resulting child is $(n+1, a_0, a_1, \dots, a_j, a_{j+1}+1, \dots, a_k)$. There are $(r-1)(a_j - a_{j-1})$ ways to do this for each j. Now the result follows. ■

Corollary 9.4 *If $\rho \in rS_k$ and $|\rho| = k(k-1)\dots21$, then*

$$|rS_n(\rho)| = \sum_{j=0}^{n} \binom{n}{j}^2 b_{j,k} j!(r-1)^j,$$

where $b_{n,k} = |S_n(k(k-1)\dots21)|$.

Proof. First suppose that $\rho = k^0(k-1)^0 \dots 2^0 1^0$. We can construct $\pi \in rS_n(\rho)$ uniquely in the following way. Let $0 \leq j \leq n$. Choose a 1-color permutation of length j avoiding $j(j-1)\dots 21$, and choose the j rows and columns for this permutation in π. Then, choose a permutation for the remaining $n-j$ rows and columns of π, and for each of these entries, choose one of the remaining $r-1$ colors. Now reindex and note that $\binom{n}{j} = \binom{n}{n-j}$, and the result follows.

For other cases, the formula follows because the generating tree for $rS_n(\rho)$ is identical to the generating tree for $rS_n(k^0(k-1)^0 \dots 2^0 1^0)$. ■

Gessel [9] uses symmetric functions to obtain generating functions for $b_{n,k}$ as determinants of matrices of Bessel functions, and then uses these generating functions to find explicit formulas for $b_{n,k}$ for small values of k. For instance, it is well-known that

$$b_{n,3} = C_n = \frac{1}{n+1}\binom{2n}{n},$$

and Gessel also shows that

$$b_{n,4} = 2\sum_{j=0}^{n} \binom{2j}{j}\binom{n}{j}^2 \frac{3j^2+2j+1-n-2jn}{(j+1)^2(j+2)(n-j+1)}.$$

Larger values of k would give much more complicated formulas for $b_{n,k}$, and Gessel does not consider these values explicitly.

10 Open questions

We prove several of our results algebraically using generating functions, but for these we have been unable to find combinatorial proofs. For many such results, the formula is simple enough that the existence of a combinatorial interpretation would not be surprising. For instance, in Theorem 3.9, we use generating trees and the kernel method to show that $B_{2k}^{180}(\bar{2}\bar{1}, \bar{2}1, 21)$ is enumerated by the central binomial coefficients. In Theorem 6.6, we use generating functions to show that $B_{2k}^{W}(\bar{1}\bar{2}, \bar{1}2, 12)$ is enumerated by the zigzag central binomial coefficients. In Theorem 2.2*(iii)*, we enumerate $B_{2k}^{180}(\bar{2}\bar{1})$ by a simple two-term recurrence relation. In this last example, the second term of the recurrence relation is subtracted, which suggests that a combinatorial argument would involve overcounting a set and then removing the objects that were overcounted.

Recent developments in tableau combinatorics may be applicable to signed permutations. Steingrímsson and Williams [19] introduced *permutation tableaux*, so named because they correspond to permutations by a bijection that nicely translates several statistics. That paper also proves results regarding equidistribution of certain *generalized* pattern statistics on permutations, by establishing a second bijection between permutation tableaux and permutations. It is natural to form *signed* permutation tableaux by giving a sign to each border segment of an unsigned permutation tableau, and it would be interesting to see if any of the results from Steingrímsson and Williams [19] can carry over to signed permutation tableaux.

Additionally, Corteel and Williams [3] introduced *staircase tableaux*, named for their staircase shape. Each box in this tableau may contain an α, β, γ or δ, or it may be left blank, according to certain rules. Corteel, Stanley, Stanton, and Williams [4] construct a bijection between staircase tableau and four-colored permutations. This bijection naturally maps a certain class of staircase tableaux—namely, those whose diagonal boxes contain only α and δ—to signed (or two-colored) permutations. A staircase tableau can be *antisymmetric*, in the sense that each α or γ is diagonally reflected to a δ or β respectively, and vice versa. Antisymmetric staircase tableaux correspond to permutations that are invariant under R_{180} before the entries are colored, but when we take colors into account there is no nontrivial symmetry under which this permutation is invariant. Perhaps this bijection can be modified such that the colors of the resulting permutation behave more nicely when the staircase tableau is antisymmetric. [There is a similar result for symmetric permutation tableaux (see [2, §2]), which does generalize easily to signed permutations.]

We give a result about pattern-restricted r-colored permutations in Corollary 9.4, but this does not involve symmetries. Perhaps some of our results for signed permutations with symmetries can be generalized to r-colored permutations with symmetries. It would be interesting to study pattern-restricted r-colored permutations that are invariant under various subgroups of the symmetry group $D_4 \oplus \mathbb{Z}_r$ or $D_4 \oplus S_r$. Admittedly, the situation becomes much more complicated with r colors—there are not only exponentially more permutations, but also more symmetries. A natural first step would be to examine r-colored permutations that are invariant under R_{180}, the reverse–complement map.

11 Acknowledgements

We would like to thank Carleton's HHMI Grant and the Carleton College Department of Mathematics for funding this work. We would also like to thank the Carleton College Department of Mathematics for giving us a newly-carpeted place to work. We would like to thank the anonymous referees for their helpful suggestions and comments. Most importantly, we would like to thank our adviser Eric Egge for spending hours guiding our research and helping us write and revise this paper.

References

[1] E. Bagno, D. Garber, and T. Mansour. Excedance number for involutions in complex reflection groups. *Sém. Lothar. Combin.*, 56: Art. B56d, 11 pp., 2007.

[2] A. Burstein. On some properties of permutation tableaux. *Ann. Comb.*, 11: 355–368, 2007.

[3] S. Corteel and L. Williams. Tableaux combinatorics for the asymmetric exclusion process and Askey–Wilson polynomials. *Duke Math. J.*, in press, arXiv:0910.1858v2.

[4] S. Corteel, R. Stanley, D. Stanton, and L. Williams. Formulae for Askey–Wilson moments and enumeration of staircase tableaux. *Trans. Amer. Math. Soc.*, in press, arXiv:1007.5174.

[5] W. M. B. Dukes and T. Mansour. Signed involutions avoiding 2-letter signed patterns. *Ann. Comb.*, 11: 387–403, 2007.

[6] W. M. B. Dukes, T. Mansour, and A. Reifegerste. Wilf classification of three and four letter signed patterns. *Discrete Math.*, 308: 3125–3133, 2008.

[7] E. S. Egge. Restricted colored permutations and Chebyshev polynomials. *Discrete Math.*, 307: 1792–1800, 2007.

[8] E. S. Egge. Restricted symmetric permutations. *Ann. Comb.*, 11: 405–434, 2007.

[9] I. M. Gessel. Symmetric functions and P-recursiveness. *J. Combin. Theory Ser. A*, 53: 257–285, 1990.

[10] O. Guibert and T. Mansour. Restricted 132-involutions. *Sém. Lothar. Combin.*, 48: Art. B48a, 23 pp., 2002.

[11] T. Mansour. Pattern avoidance in coloured permutations. *Sém. Lothar. Combin.*, 46: Art. B46g, 12 pp., 2001.

[12] T. Mansour and Y. Sun. Even signed permutations avoiding 2-letter signed patterns. *J. Math. Res. Exposition* 29: 813–822, 2009.

[13] T. Mansour and Y. Sun. Excedance numbers for permutations in complex reflection groups. *Sém. Lothar. Combin.*, 58: Art. B58b, 7 pp., 2008.

[14] T. Mansour and J. West. Avoiding 2-letter signed patterns. *Sém. Lothar. Combin.*, 49: Art. B49a, 11 pp., 2002.

[15] T. Muir. On self-conjugate permutations. *Proc. Roy. Soc. Edinburgh*, 17: 7–22, 1889.

[16] R. Simion. Combinatorial statistics on type-B analogues of noncrossing partitions and restricted permutations. *Electron. J. Combin.*, 7: #R9, 27 pp., 2000.

[17] R. Simion and F. W. Schmidt. Restricted permutations. *European J. Combin.*, 6: 383–406, 1985.

[18] N. J. A. Sloane. *The On-Line Encyclopedia of Integer Sequences*, published electronically at `http://oeis.org`, 2011.

[19] E. Steingrímsson and L. K. Williams. Permutation tableaux and permutation patterns. *J. Combin. Theory Ser. A*, 114: 211–234, 2007.

[20] J. West. Generating trees and forbidden subsequences. *Discrete Math.*, 157: 363–374, 1996.