Baxter Permutations, Snow Leopard Permutations, and Restricted Catalan Paths

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Complete Baxter Permutations

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A permutation is called *Baxter* if it avoids the generalized patterns $3-14-2$ and $2-41-3$.

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π is a complete Baxter permutation if for all i with $1 \leq i \leq |\pi|$:

- $\pi(i)$ is even if and only if i is even
- if $\pi(x) = i$, $\pi(z) = i+1$, and y is between x and z, then $\pi(y) < i$ if i is even and $\pi(y) > i+1$ if i is odd

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Example

3 2 1 4 13 12 7 8 11 10 9 6 5

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				13	12	7	8	11	10	9	6	
3	2	1	4									5

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Odd entries:

- Determine a complete Baxter permutation
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A permutation is called *anti-Baxter* if it avoids the generalized patterns $3-41-2$ and $2-14-3$.

Compatibility

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If there exists a complete Baxter permutation π such that π_1 and π_2 are the permutations induced on the odd and even entries of π, respectively, we say that π_1 and π_2 are *compatible*.

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Baxter permutations compatible with unique anti-Baxter permutation

$$1 \quad 4 \quad 3 \quad 2 \quad 5$$

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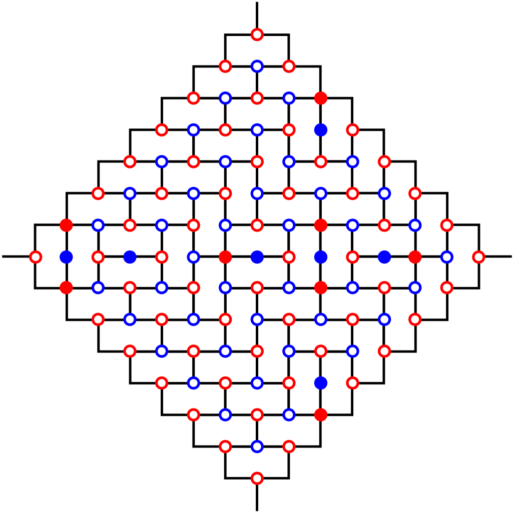
Anti-Baxter permutation may be compatible with multiple Baxter permutations

$$127654389$$
$$125674389$$
$$127634589$$

Aztec Diamond



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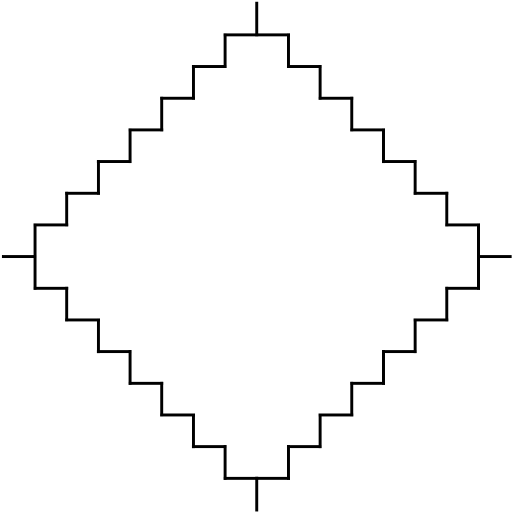

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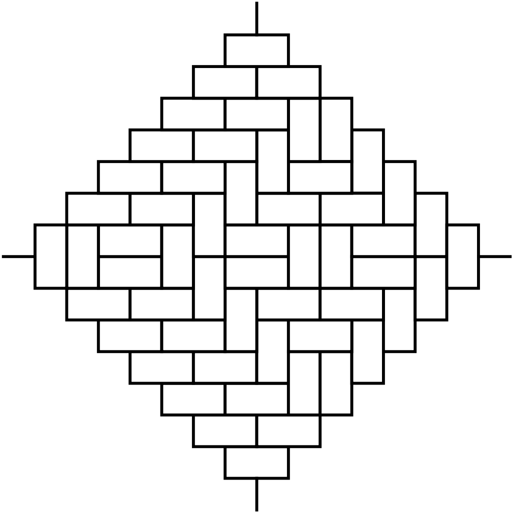

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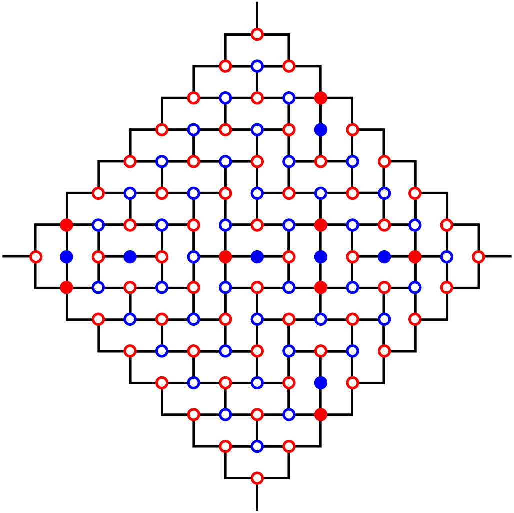

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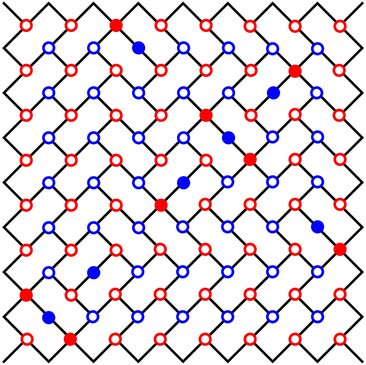

DABPs

Doubly Alternating Baxter Permutations

- ascents and descents alternate in π, beginning with ascent
- ascents and descents alternate in π^{-1}, beginning with ascent
- Baxter

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Theorem [Guibert & Linusson, 2000]

The number of DABPs of length $2n$ is C_n, the n^{th} Catalan number.

Snow Leopard Permutations

Definition

We call the permutations of length n which are compatible with the DABPs of length $n+1$ the *snow leopard permutations*.

Examples

1
123, 321
12345, 14325, 34521, 54123, 54321

Properties

- anti-Baxter
- identity and reverse identity are always snow leopard
- odd entries in odd positions, even entries in even positions

Decomposition of SLPs

Theorem [Caffrey, Egge, Michel, Rubin, Ver Steegh]

A permutation π of length $2n$ is an SLP if and only if there exists an SLP σ of length $2n-1$ such that $\pi = 1 \oplus \sigma^c$.

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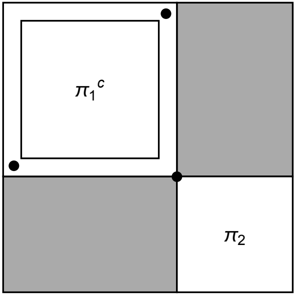
A permutation π is an SLP if and only if there exist SLPs π_1 and π_2 such that $\pi = (1 \oplus \pi_1^c \oplus 1) \ominus 1 \ominus \pi_2$.



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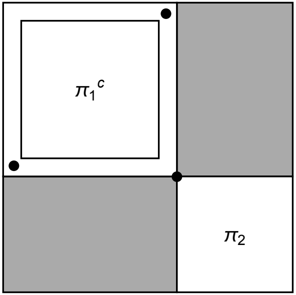

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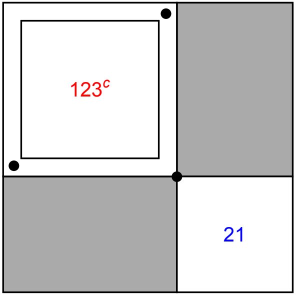

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$$(1 \oplus 123^c \oplus 1) \ominus 1 \ominus 21$$

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Theorem [Caffrey, Egge, Michel, Rubin, Ver Steegh]

$SL_n :=$ the set of snow leopard permutations of length $2n-1$

- $|SL_1| = 1, \ |SL_2| = 2$
- $|SL_{n+1}| = \sum_{j=0}^{n} |SL_j||SL_{n-j}|$
- $|SL_n| = C_n$

Bijection with Catalan paths

$$\begin{bmatrix} & 3 & 6 & 5 & 4 & 7 & 2 & 1 & \\ & & & & & & & & \end{bmatrix}$$

Bijection with Catalan paths

8 3 6 5 4 7 2 1 0

[ ]

Bijection with Catalan paths

$$\begin{array}{c} 8 \qquad 3 \qquad 6 \qquad 5 \qquad 4 \qquad 7 \qquad 2 \qquad 1 \qquad 0 \\ \begin{bmatrix} d & a & d & d & a & d & d & d \\ N & E & N & N & E & N & N & N \end{bmatrix} \end{array}$$

Bijection with Catalan paths

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8 \qquad 3 \qquad 6 \qquad 5 \qquad 4 \qquad 7 \qquad 2 \qquad 1 \qquad 0 \\
\begin{bmatrix} & d & a & d & d & a & d & d & d & \\ & N & {\color{red}E} & N & {\color{red}N} & E & {\color{red}N} & N & {\color{red}N} & \end{bmatrix}
\end{array}$$

Bijection with Catalan paths

$$\begin{array}{ccccccccc} 8 & 3 & 6 & 5 & 4 & 7 & 2 & 1 & 0 \end{array}$$

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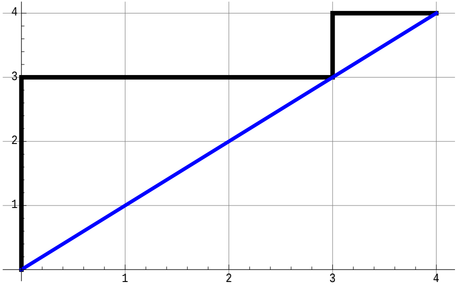


Odd and Even Knots

Definition

We call the permutation induced on the even entries of an SLP π an *even knot* ($\text{even}(\pi)$) and the permutation induced on the odd entries an *odd knot* ($\text{odd}(\pi)$).

Examples

Odd knots: $\emptyset$, 1, 12, 21, 123, 231, 321, 321
Even knots: $\emptyset$, 1, 12, 21, 123, 132, 213, 231, 312, 321

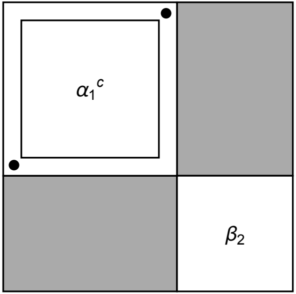
Decomposition of Even and Odd Knots

Odd knot β

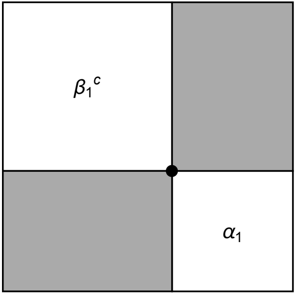
$\beta = (1 \oplus \alpha_1^c \oplus 1) \ominus \beta_1$ for odd knot β_1 and even knot α_1.

Even knot α

$\alpha = \beta_1^c \ominus 1 \ominus \alpha_1$ for odd knot β_1 and even knot α_1.



(a) Odd knot decomposition



(b) Even knot decomposition

What are the odd and even knots counted by?

n	0	1	2	3	4	5	6
$\lvert EK_n \rvert$	1	1	2	6	17	46	128
$\lvert OK_n \rvert$	1	1	2	4	9	23	63

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Theorem [Egge, Rubin]

The odd knots of length n are in bijection with the set of Catalan paths of length n which do not contain NEEN.

Theorem [Egge, Rubin]

The even knots of length n are in bijection with the set of Catalan paths of length $n+1$ which have no ascent of length exactly 2.

Entangled Knots

Definition

We say that an even knot α and an odd knot β are *entangled* if there exists an SLP π such that $\text{even}(\pi) = \alpha$ and $\text{odd}(\pi) = \beta$.

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Theorem [Egge, Rubin]

The even knots of length $n-1$ entangled with the identity permutation of length n are the 3412-avoiding involutions of length $n-1$.

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Theorem [Egge, Rubin]

The odd knots of length $n+1$ entangled with the reverse identity permutation of length n are the complements of the 3412-avoiding involutions of length $n+1$.

Entangled Knots

Corollary [Egge, Rubin]

The number of even knots of length $n-1$ entangled with the identity permutation of length n is M_{n-1}, where M_n is the n^{th} Motzkin number.

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Conjecture

For each even (resp. odd) knot, the number of entangled odd (resp. even) knots is a product of Motzkin numbers.

Open Questions

- Is it true that for each knot the number of entangled knots is a product of Motzkin numbers?

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- Can every knot be constructed from 3412-avoiding involutions using just basic permutation constructions?
- Are there relationships between natural statistics on lattice paths and natural statistics on snow leopard permutations, even knots, or odd knots?

References

B. Caffrey, E. S. Egge, G. Michel, K. Rubin, & J. Ver Steegh
Domino Tilings of Aztec Diamonds, Baxter Permutations and Snow Leopard Permutations
Involve, pending

H. Canary
Aztec Diamonds and Baxter Permutations
Electron. J. Combin.,17(1), 2010

F. R. K. Chung, R. L. Graham, V. E. Hoggatt Jr., & M. Kleiman.
The Number of Baxter Permutations.
J. Combin. Theory Ser. A, 24(3):382–394, 1978.

O. Guibert.
Combinatoire des permutations à motifs exclus en liaison avec mots, cartes planaires et tableaux de Young.
PhD thesis, Université Bordeaux I, 1995

O. Guibert & S. Linusson
Doubly Alternating Baxter Permutations are Catalan
Discrete Math, 217(1–3):157–166, 2000.