Symmetric Permutations Avoiding a Pattern of Length Three and a Pattern of Length Four

David Lonoff and Jonah Ostroff
Carleton College
Northfield, MN 55057 USA

June 5, 2008

Abstract

Symmetric pattern-avoiding permutations are restricted permutations which are invariant under actions of certain subgroups of D_4, the symmetry group of a square. We examine pattern-avoiding permutations with 180° rotational-symmetry. In particular, we use combinatorial techniques to enumerate symmetric permutations which avoid one pattern of length three and one pattern of length four. The resulting sequences involve well-known sequences such as the alternating Fibonacci numbers, Catalan numbers, triangular numbers, and powers of two.

Keywords: Fibonacci identity, pattern-avoiding permutation, restricted permutation, signed permutation, symmetric permutation.

1 Introduction and Notation

The *one-line notation* form of a permutation π of $[n]$, where $[n] = \{1, 2, \ldots, n\}$, is written $\pi(1)\pi(2)\ldots\pi(n)$. For instance, if π is a permutation of [5] whose cyclic form is (13), then π's one-line notation form is 32145. In this paper, we will use S_n to refer to the set of permutations of $[n]$ written in one-line notation.

If $\pi \in S_n$ and $\sigma \in S_k$, then π *contains σ as a pattern* if some subsequence of π of length k has the same relative order as σ. For instance, since 5287 has the same relative order as 2143, the permutation 13524867 contains 2143. We say that π *avoids* σ whenever π does not contain σ. For some set R of permutations (not necessarily of the same length), we let $S_n(R)$ denote the set of permutations of length n which avoid every pattern in R. Often braces are omitted when the elements of R are included in this notation. For instance, 13524867 has no decreasing subsequence of length 3, so $13524867 \in S_8(321)$, but it is not an element of $S_8(2143)$. We will often refer to the elements of $S_n(R)$ as *pattern-avoiding permutations*, and the elements of R as *forbidden patterns*. If π contains σ, then the subsequence of π with the same relative order as σ is called a σ *subsequence*.

The *diagram* of a permutation $\pi \in S_n$ is formed by creating an $n \times n$ grid whose rows and columns are labeled from 1 to n from bottom to top and left to right. A dot is placed in the cell (i, j) exactly when $\pi(i) = j$. It is easy to see that $\pi \in S_n$ contains $\sigma \in S_k$ exactly

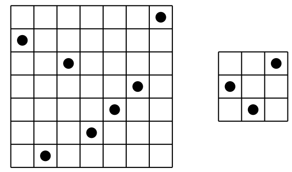


Figure 1: 6152347 contains 213.

if we can choose some k rows and k columns from the diagram of π, the intersection points of which form the diagram of σ. As the diagram in Figure 1 illustrates, 6152347 contains 213, since the intersections of the first, fifth and seventh columns with the third, sixth and seventh rows form the diagram of 213.

The representation of a permutation on a square further motivates three common operations on permutations. For $\pi \in S_n$, the *reverse* of π is the permutation $\pi^r = \pi(n)\pi(n-1)\dots\pi(2)\pi(1)$. That is, π^r is the permutation whose diagram is obtained by reflecting the diagram of π over a vertical axis. Similarly, the *complement* of π is the permutation whose entries follow the formula $\pi^c(j) = n + 1 - \pi(j)$. The diagram of π^c is obtained from that of π by reflection over a horizontal axis. Finally, the *inverse* of π, which we denote by π^i, is the inverse of π as a function, so if $\pi(j) = k$, then $\pi^i(k) = j$. The diagram of π^i is that of π, reflected over the diagonal from the lower left corner to the upper right corner.

This set of operations, when considered as symmetries of a square, motivates a brief foray into the algebra of dihedral groups. D_4, the group of the eight symmetries of a square, is well-known to be generated by the above mappings r, c, and i. We say that a permutation is *preserved* under some symmetry $g \in D_4$ if its diagram is unchanged by g. Equivalently, if we consider D_4 to be a group of actions on the set of diagrams of permutations in S_n, then $\pi \in S_n$ is preserved by $g \in D_4$ if g is in the stabilizer of the diagram of π. Since the stabilizer of a diagram is a subgroup of D_4, we can consider the possible symmetries of a permutation by considering the 10 distinct subgroups of D_4.

Of course, many of these subgroups will be uninteresting to consider. For instance, for $n \geq 2$, no permutation in S_n is preserved by r, since the first and last elements in one-line notation are never equal. There are four subgroups which are interesting to study, and have been to various extents (a sample of the literature includes [2, 3, 4, 5]). In this paper we will focus our attention on the subgroup $\{e, rc\}$, and label the set of permutations of length n preserved by this subgroup S_n^{rc}. This is the set of permutations whose diagrams are symmetric under a 180° rotation. For instance, $412563 \in S_6^{rc}$. Just as in S_n, we use $S_n^{rc}(R)$ to refer to the set of permutations of length n whose diagrams are symmetric under 180° rotation and which avoid every pattern in R. These sets have been enumerated for certain R by Egge [3], in particular for all $R \subseteq S_3$. In this paper we will enumerate $S_n^{rc}(\sigma, \tau)$ for all $\sigma \in S_3$ and $\tau \in S_4$.

We begin by counting S_n^{rc} itself. To this end, we introduce B_n, the set of *signed permutations of length n*. An element $\pi \in B_n$ is a permutation of $[n]$ written in one-line notation in which each entry of π may or may not have a bar over it. We note that S_n appears in B_n as the set of signed permutations without bars. Since there are 2^n choices for bar placement and $n!$ arrangements $[n]$, it is easy to see that $|B_n| = 2^n n!$.

From this point forward in the paper, it will be much more common to refer to S^{rc}_{2n} and S^{rc}_{2n+1} than simply S^{rc}_n, since many arguments revolve heavily around parity. We now claim that $|B_n| = |S^{rc}_{2n}| = |S^{rc}_{2n+1}|$, and we shall prove this through two bijections.

We first define $s : B_n \mapsto S^{rc}_{2n}$ by observing that a permutation in S^{rc}_{2n} is in fact determined by its first n entries. We further observe that any of the first n entries of $\rho \in S^{rc}_{2n}$ may be in either the top-left or bottom-left quadrants, and that no row occupied in the top-left quadrant may be the image under c of a row occupied in the bottom-left. This motivates the following definition.

Definition 1.1 *For $\pi \in B_n$, we define $\pi^s \in S^{rc}_{2n}$ as follows.*

$$\pi^s(j) = \begin{cases} \pi(j) & \text{if } \pi(j) \text{ is barred and } 1 \le j \le n \\ 2n+1-\pi(j) & \text{if } \pi(j) \text{ is unbarred and } 1 \le j \le n \\ 2n+1-\pi(2n+1-j) & \text{if } \pi(2n+1-j) \text{ is barred and } n+1 \le j \le 2n \\ \pi(2n+1-j) & \text{if } \pi(2n+1-j) \text{ is unbarred and } n+1 \le j \le 2n \end{cases}$$

Graphically, π^s is the permutation whose diagram is obtained by placing the barred entries of π in the lower-left quadrant of a $2n \times 2n$ grid, placing the unbarred entries in the top-left quadrant after reflection over the horizontal, and filling out the top-right and bottom-right quadrants symmetrically. We next define $u : S^{rc}_{2n} \mapsto S^{rc}_{2n+1}$ by observing that for any $\rho \in S^{rc}_{2n+1}$, we have $\rho(n+1) = n+1$, since $\rho(n+1) = 2n+2-\rho((2n+2)-(n+1))$. Then all that is left is to determine the other $2n$ rows and columns.

Definition 1.2 *For $\pi \in S^{rc}_{2n}$, we define $\pi^u \in S^{rc}_{2n+1}$ as follows.*

$$\pi^u(j) = \begin{cases} \pi(j) & \text{if } 1 \le j \le n \text{ and } 1 \le \pi(j) \le n \\ \pi(j)+1 & \text{if } 1 \le j \le n \text{ and } n+1 \le \pi(j) \le 2n \\ n+1 & \text{if } j = n+1 \\ \pi(j-1) & \text{if } n+2 \le j \le 2n+1 \text{ and } 1 \le \pi(j) \le n \\ \pi(j-1)+1 & \text{if } n+2 \le j \le 2n+1 \text{ and } n+1 \le \pi(j) \le 2n \end{cases}$$

We claim without proof that both s and u are bijections. A more rigorous definition and a proof may be found in [3, Sec. 2]. In this paper, we will rarely refer to u, and will instead refer to $t : B_n \mapsto S^{rc}_{2n+1}$ defined by $t = u \circ s$.

In this paper, we group our results by proof technique. In section 2, we consider the conditions on $\pi \in B_n$ such that π^s and π^t avoid some pattern σ. These results will often involve reducing pattern avoidance to a question of which entries are barred, with the remaining entries of the permutation mostly determined by the answer to this question. This method is particularly helpful when the forbidden patterns are themselves symmetric under rc, and can thus be represented as elements of B_k for various k.

We first take advantage of the restrictions imposed by the fact that $\pi^t(n+1) = n+1$ to show that

$$|S^{rc}_{2n+1}(123, 1432)| = |S_n(321, 4123, 2341)| = a_n \qquad (n \ge 1),$$

where a_n is the sequence A116716 in Sloane [6].

By using a result from West [7], we next show that

$$|S^{rc}_{2n+1}(123, 2413)| = F_{2n-2} \qquad (n \ge 1)$$

where F_n is the n-th Fibonacci number, defined by $F_0 = 1$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Then, by asking which entries are barred and using an apparently new Fibonacci identity, we show that

$$|S_{2n}^{rc}(123, 2413)| = F_{2n} \qquad (n \geq 0).$$

We then use a similar bar-chasing technique and a result from Simion and Schmidt [5] to find that

$$|S_{2n}^{rc}(123, 4231)| = n^2 + 1 \qquad (n \geq 0)$$

and

$$|S_{2n+1}^{rc}(123, 4231)| = \binom{n}{2} + 1 \qquad (n \geq 0).$$

A similar argument also shows that

$$|S_{2n}^{rc}(123, 3412)| = 2^{n+1} - (n+1) \qquad (n \geq 0)$$

and

$$|S_{2n+1}^{rc}(123, 3412)| = 1 \qquad (n \geq 0).$$

In section 3, we enumerate permutations based on the position of the 1. Since this also determines the position of $2n$ (or $2n+1$), conditioning on this value (strictly speaking, $\pi^{-1}(1)$) often produces simple information about the rest of the permutation, sometimes inductively. We start out with an elementary argument that

$$|S_{2n}^{rc}(123, 4312)| = 6 \qquad (n \geq 2)$$

and

$$|S_{2n+1}^{rc}(123, 4312)| = 1 \qquad (n \geq 0).$$

We then use somewhat more complicated counting to show that

$$|S_{2n}^{rc}(123, 2431)| = F_{n+2} + 1 \qquad (n \geq 2)$$

and

$$|S_{2n+1}^{rc}(123, 2431)| = F_{n+1} - 1 \qquad (n \geq 1).$$

In the appendix, we prove the new Fibonacci identity used in Section 2, and we provide a natural generalization.

2 Where the Wild Bars Are

Recall from the Introduction that for any $n \geq 0$ and any set R of forbidden patterns we write $S_n^{rc}(R)$ to denote the set of permutations in S_n which are invariant under rc and which avoid every pattern in R. In this section we demonstrate one proof technique that involves determining conditions on $\pi \in B_n$ such that $\pi^s \in S_{2n}^{rc}(R)$ or $\pi^t \in S_{2n+1}^{rc}(R)$. Often, we begin by asking which entries are barred and where they are located. In Theorems 2.6 and 2.8, knowing which entries are barred determines most of the information about π. In Theorems 2.14, 2.15, 2.20, and 2.21, this information completely determines π.

It will be helpful to develop some notation for dealing with elements of B_n whose images under s or t avoid certain patterns. For $n \geq 0$ and some set R of patterns, we will let $B_n^s(R)$ denote the set of signed permutations of length n whose images under s avoid every pattern in R. $B_n^t(R)$ is defined similarly. Also, we say that some $\pi \in B_n$ *contains* $\sigma \in B_k$ *as a pattern* whenever some subsequence of π has the same length and pairwise comparisons as σ, and corresponding entries in both sequences are barred. For instance, if $\pi = \bar{3}51\bar{2}4$ then π contains $1\bar{2}3$ as a pattern but not 123. σ is a *forbidden pattern of* $B_n^s(R)$ if whenever $\pi \in B_n$ contains σ as a pattern, π^s contains some pattern from R. A forbidden pattern of $B_n^t(R)$ is defined analogously. Also, for $\pi \in B_n$, we define $bar(\pi)$ to be the set of elements which are barred in π. From our earlier example, $bar(\pi) = \{2, 3\}$.

For our first results, we show that counting permutations in $S_{2n+1}^{rc}(R)$ can be reduced to known enumerative results when R contains permutations of a certain type which we now define.

Definition 2.1 *We say a permutation $\pi \in S_n$ is* skew decomposable at index k *whenever there exists an index k such that for all $1 \leq i \leq k < j \leq n$, we have $\pi(i) > \pi(j)$. We say π is* skew indecomposable *whenever π cannot be skew decomposed. Furthermore, if π is skew decomposable at index k, then we say $\pi = \sigma \ominus \tau$, where σ is the relative order of $\pi(1)\pi(2)\ldots\pi(k)$ and τ is the relative order of $\pi(k+1)\pi(k+2)\ldots\pi(n)$. We call σ and τ the* summands *of π.*

Example 2.2 *$\pi = 35412 = 132 \ominus 12$ is skew decomposable at index* 3. 2413 *is skew indecomposable.*

35412   2413

The following lemma by Egge ([3, Lem 2.11]) will be quite useful in this section, as we are dealing with symmetric permutations which avoid sets including 123. We use it to prove Lemma 2.4 which is a slightly stronger case.

Lemma 2.3 *If $123 \in R$ and $\pi \in B_n^t(R)$, then $bar(\pi) = \emptyset$.*

Proof. The negation of this lemma implies there exists $k \in bar(\pi)$. But $k, n+1, 2n+2-k$ is a 123 subsequence. □

Lemma 2.4 *Suppose R is a set of skew indecomposable permutations. For all $\pi \in B_n$, the following are equivalent.*

(i) $\pi^t \in S_{2n+1}^{rc}(123 \cup R)$.

(ii) $\pi \in S_n(321 \cup R^c \cup R^r)$*, where* $R^c = \{\sigma^c | \sigma \in R\}$ *and* $R^r = \{\sigma^r | \sigma \in R\}$.

Proof. (i) $\Longrightarrow$ (ii) Suppose $\pi \in B_n$ and $\pi^t \in S_{2n+1}^{rc}(123 \cup R)$. Then by Lemma 2.3 we have $bar(\pi) = \emptyset$, which implies $\pi \in S_n(321)$. If π contains $\sigma^c \in R^c$ then by construction π^t will contain $(\sigma^c)^c = \sigma \in R$, which is forbidden. Similarly, if π contains $\sigma^r \in R^r$, then π^t contains σ^{rc}. But π^t is invariant under rc, so π^t also contains $(\sigma^{rc})^{rc} = \sigma$, which is forbidden.

(ii) $\Longrightarrow$ (i) Suppose π^t contains $\sigma \in 123 \cup R$. Since $\pi \in S_n$, every entry to the left of $\pi^t(n+1) = n+1$ is larger than every entry to the right of $n+1$. This implies that π^t is skew decomposable. Since σ is skew indecomposable, either σ is contained entirely to the left or entirely to the right of $n+1$. If σ is entirely to the left of $n+1$, then by construction π contains σ^c. If σ is entirely to the right of $n+1$, then σ^{rc} (the rotated image of σ) is entirely to the left of $n+1$. In this case π contains $(\sigma^{rc})^c = \sigma^r$. $\square$

With these lemmas in hand, we show how enumerating $S^{rc}_{2n+1}(123, 1432)$ is easily reduced to a known result.

Theorem 2.5 *For all* $n \geq 1$*,*

$$|S^{rc}_{2n+1}(123, 1432)| = |S_n(321, 4123, 2341)| = a_n, \tag{1}$$

where a_n *is the sequence A116716 in Sloane [6], with generating function* $\frac{x(x+1)(x^3-2x^2+x-1)}{(x^2+1)(x^3-x^2-2x+1)}$.

Proof. Because 1432 is skew indecomposable, Lemma 2.4 tells us that t is a bijection from $S_n(321, 4123, 2341)$ to $S^{rc}_{2n+1}(123, 1432)$, and (1) follows from Pudwell's enumeration of this set. $\square$

We note that we have no enumeration for $S^{rc}_{2n}(123, 1432)$. This remains an open problem. Next, we enumerate $S^{rc}_n(123, 2413)$ using a similar technique.

Theorem 2.6 *For all* $n \geq 1$*,*

$$|S^{rc}_{2n+1}(123, 2413)| = F_{2n-2}. \tag{2}$$

Proof. Note that 2413 is skew indecomposable. So by Lemma 2.4, t is a bijection from $S_n(321, 3142)$ to $S^{rc}_{2n+1}(123, 2413)$. Thus $|S^{rc}_{2n+1}(123, 2413)| = |S_n(321, 3142)|$, and (2) follows from [7, Table 1]. $\square$

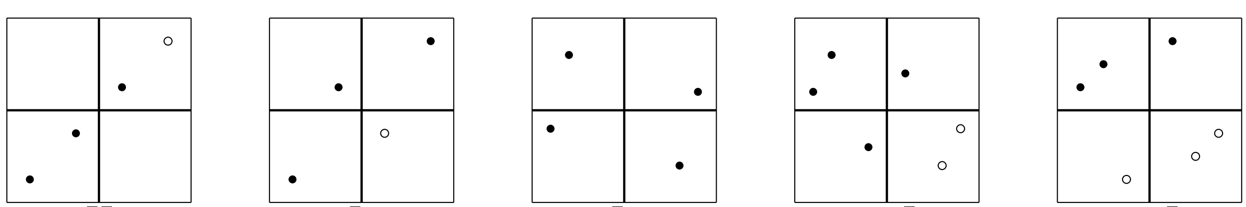
The even case is a bit more difficult, so we first prove the following lemma.

Lemma 2.7 *Fix* $n \geq 0$ *and* $\pi \in B_n$*. Then the following are equivalent.*

(i) $\pi^s \in S^{rc}_{2n}(123, 2413)$

(ii) π *avoids* 321, 3142, $\bar{1}\bar{2}$, $\bar{1}2$, $\bar{2}1$, $31\bar{2}$, $32\bar{1}$.

Proof. (i) $\Longrightarrow$ (ii) Note that $321^c = 123$ and $3142^c = 2413$ so clearly π must avoid these patterns. For the remaining arguments, see the accompanying figures. In each figure, a pattern and its image under rc are shown. The filled in circles indicate the occurrence of a forbidden pattern.



$\bar{1}\bar{2}$ $\quad$ $\bar{1}2$ $\quad$ $\bar{2}1$ $\quad$ $31\bar{2}$ $\quad$ $32\bar{1}$

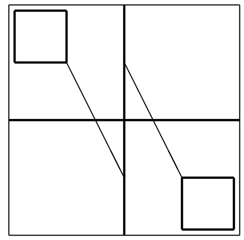


Figure 2: The general form of $\pi^s \in S_{2n}^{rc}(123, 2413)$

(ii) $\Longrightarrow$ (i) To show that avoiding the above patterns is sufficient, we show that π^s must be of the form shown in Figure 2. In this diagram, the boxes represent subpermutations which avoid 123 and 2413 and the parallel lines mean that a dot in a given row could be located on one line or the other.

First, if $bar(\pi) = \emptyset$, then $\pi \in S_n(321, 3142)$ and the square subpermutation fills the entire left quadrant in the diagram above. Otherwise, $bar(\pi)$ is non-empty, and it contains a smallest element, call it k. Then $bar(\pi)$ is a freely chosen subset of $\{k, k+1, \ldots, n\}$. We have shown that π avoids $\bar{1}2$, $\bar{1}\bar{2}$, and $\bar{2}1$, which implies that all of the elements of $bar(\pi)$ must appear at the right end of π and in decreasing order. Furthermore, the unbarred elements of $\{k+1, k+2, \ldots, n\}$ must appear in ascending order immediately preceding the barred elements, or else we get a $32\bar{1}$ or $31\bar{2}$, which are also forbidden. Finally, the elements in $[k-1]$ need to be arranged to avoid 321 and 3142. These conditions, along with rc-symmetry, do indeed force π^s into the above form. One can easily check that a permutation of the above form avoids 123 and 2413, and we are done. $\square$

Theorem 2.8 *For all $n \geq 0$,*

$$|S_{2n}^{rc}(123, 2413)| = F_{2n}. \tag{3}$$

Proof. As we saw in Lemma 2.7, if $\pi^s \in S_{2n}^{rc}(123, 2413)$ and $bar(\pi) = \emptyset$, then $\pi \in S_n(321, 3142)$, and this yields F_{2n-2} permutations by [7, Table 1]. As before, if $bar(\pi)$ is non-empty, let k be its smallest element. Then $bar(\pi) \subseteq \{k, k+1, \ldots, n\}$ which can be chosen in 2^{n-k} ways. The proof of Lemma 2.7 tells us that everything is determined besides the order of $[k-1]$, which must be arranged to avoid 321 and 3142. This can be done in $F_{2(k-1)-2} = F_{2k-4}$ ways (again by [7, Table 1]). Summing over all positive k, we get

$$|S_{2n}^{rc}(123, 2413)| = F_{2n-2} + \sum_{k=1}^{n} F_{2k-4} 2^{n-k}$$

which evaluates to F_{2n}, a result which we prove in the Appendix. $\square$

Next, we turn our attention to $S_n^{rc}(123, 4231)$, which is determined solely by which entries in π are barred.

Lemma 2.9 *If $\pi \in B_n^s(123, 4231)$, then $bar(\pi)$ consists of a single (possibly empty) set of consecutive integers.*

Proof. Suppose not, so there exists $i < j < k$ such that $i, k \in bar(\pi)$, $j \notin bar(\pi)$. If $\overline{k}$ precedes j, then k $2n+1-j$ $2n+1-i$ is a 123 subsequence. If j precedes $\overline{k}$, then $2n+1-j$ k $2n+1-k$ j is a 4231 subsequence. $\square$

Lemma 2.10 *The following are all forbidden patterns of $B_n^s(123, 4231)$.*

(i) $\bar{1}\,\bar{2}$

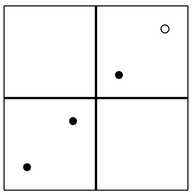
(ii) $1\,\bar{2}$

(iii) $\bar{1}\,2$

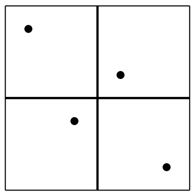
(iv) $\bar{3}\,2\,1$

(v) $3\,2\,\bar{1}$

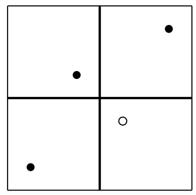
Proof. As in the proof of Lemma 2.7, the proof of each is shown in the accompanying figures.



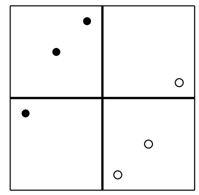
$\bar{1}\bar{2}$



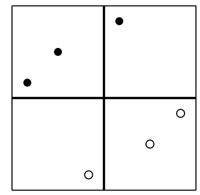
$1\bar{2}$



$\bar{1}2$



$\bar{3}21$



$32\bar{1}$ $\square$

Lemma 2.11 *If $\pi \in B_n^s(123, 4231)$ and $bar(\pi) \neq \emptyset$, then π is uniquely determined by $min(bar(\pi))$ and $max(bar(\pi))$, where $1 \leq min(bar(\pi)) \leq max(bar(\pi)) \leq n$.*

Proof. By Lemma 2.9, $bar(\pi) = \mathbb{N} \cap [min(bar(\pi)), max(bar(\pi))]$. By Lemma 2.10 (i), the elements of $bar(\pi)$ appear in π in decreasing order. By (ii), the elements less than $min(bar(\pi))$ appear after $min(bar(\pi))$, and by (iv) they appear in ascending order. By (iii), the elements greater than $max(bar(\pi))$ appear before $max(bar(\pi))$, and by (v) they appear in ascending order. $\square$

Lemma 2.12 *For all $n \geq 1$, we have*

$$|B_n^s(123, 4231) \setminus S_n| = \binom{n+1}{2}. \tag{4}$$

Proof. There are $\binom{n+1}{2}$ ways to pick $min(bar(\pi))$ and $max(bar(\pi))$. By Lemma 2.11, this forces the rest of the permutation. The result will be of the form shown in Figure 3, which it is easy to see avoids both 123 and 4231. $\square$

Lemma 2.13 *For all $n \geq 1$,*

$$B_n^s(123, 4231) \cap S_n = S_n(321, 132). \tag{5}$$

Proof. ($\subseteq$) 321^s contains a 123 subsequence, and 132^s contains a 4231 subsequence.

($\supseteq$) $S_n(321, 132) \subseteq S_n$ is trivial. Furthermore, suppose $\pi \in S_n(321, 132)$ and $\pi \notin B_n^s(123, 4231)$, so π contains a 123 or 4231 subsequence. Since both 123 and 4231 are 180°-symmetric, we may assume that at least half of either sequence falls in the first half π^s, and in particular the upper-left quadrant (since $bar(\pi) = \emptyset$). If the 12 of the 123 falls in the upper left, then so does the 3, since it cannot be less than the 2. This implies that π

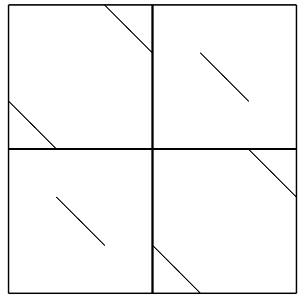


Figure 3: The general form of $\pi \in S_{2n}^{rc}(123, 4231)$.

contains 321. Likewise, if the 42 of the 4231 appears in the upper left, then so does the 3, but 423 in π^s corresponds to a 132 subsequence in π. $\square$

In view of (5) it is useful to recall that $|S_n(132, 321)| = \binom{n}{2} + 1$, which was first proved by Simion and Schmidt [5, Prop. 11].

Theorem 2.14 *For all $n \geq 0$,*

$$|S_{2n}^{rc}(123, 4231)| = n^2 + 1. \tag{6}$$

Proof. We have

$$\begin{aligned} |S_{2n}^{rc}(123, 4231)| &= |B_n^s(123, 4231)| \\ &= |B_n^s(132, 4231) \setminus S_n| + |B_n^s(132, 4231) \cap S_n| \\ &= \binom{n+1}{2} + \binom{n}{2} + 1 \qquad \text{(by (4) and [5, Prop. 11])} \\ &= n^2 + 1. \end{aligned}$$

$\square$

Theorem 2.15 *For all $n \geq 0$,*

$$|S_{2n+1}^{rc}(123, 4231)| = \binom{n}{2} + 1. \tag{7}$$

Proof. This is immediate from Lemma 2.3, equation (5), and [5, Prop. 11]. $\square$

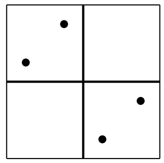
Next, we turn to $S_n^{rc}(123, 3412)$.

Lemma 2.16 *The following are forbidden patterns of $B_n^s(123, 3412)$.*

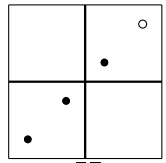
(i) 2 1

(ii) $\overline{1}\ \overline{2}$

Proof. The proof is shown in the accompanying diagrams.



21



$\bar{1}\bar{2}$

□

Lemma 2.17 *If $\pi \in B_n^s(123, 3412), i < j, i \in bar(\pi)$, and $j \notin bar(\pi)$, then j precedes every barred entry in π.*

Proof. Suppose some $\bar{k}$ precedes j. Then k, $2n+1-j$, $2n+1-i$ is a 123 subsequence. □

Lemma 2.18 *If $\pi \in B_n^s(123, 3412)$ and $bar(\pi)$ is not of the form $[n] \setminus [k]$ for any $0 \le k \le n$, then π is completely determined by $bar(\pi)$.*

Proof. Since $bar(\pi)$ is not of the above form, the hypothesis of Lemma 2.17 is true for $j = max([n] \setminus bar(\pi))$ and $i = min(bar(\pi))$. So j must precede every barred entry in π. Furthermore, by Lemma 2.16, the unbarred entries must appear in ascending order and the barred entries must appear in descending order. Thus j must appear last among the unbarred entries, so all unbarred entries must precede all barred entries. □

Lemma 2.19 *If $\pi \in B_n^s(123, 3412)$ and $bar(\pi)$ is of the form $[n] \setminus [k]$ for some $0 \le k \le n$, then π is determined by the choice of which positions in π are unbarred and which are barred.*

Proof. Once it is determined where the barred and unbarred entries go, then it is simply a matter of placing the lowest k numbers in ascending order in the unbarred positions and the highest $n-k$ terms in descending order in the unbarred positions. This is sufficient for π^s to avoid 123 and 3412. □

Theorem 2.20 *For all $n \ge 0$, we have*

$$|S_{2n}^{rc}(123, 3412)| = 2^{n+1} - (n+1). \tag{8}$$

Proof. We prove (8) by counting the choices for $\pi \in B_n^s(123, 3412)$. From Lemma 2.18, there are 2^n choices for $bar(\pi)$, but $n+1$ of them are of the form $[n] \setminus [k]$. Those of such a form are determined instead by the 2^n choices for which positions in π are barred. In total, then, $|S_{2n}^{rc}(123, 3412)| = 2^n - (n+1) + 2^n = 2^{n+1} - (n+1)$. □

Theorem 2.21 *For all $n \ge 0$,*

$$|S_{2n+1}^{rc}(123, 3412)| = 1. \tag{9}$$

In particular, if $\pi \in S_{2n+1}^{rc}(123, 3412)$ then $\pi = 2n{+}1\ 2n\ 2n{-}1 \ldots 2\ 1$.

Proof. By Lemma 2.3, no entries of $\pi \in B_n^t(123, 3412)$ may be barred, and by the same argument from Lemma 2.16(i) all unbarred entries must appear in ascending order. This leaves exactly one choice for π^t, namely the permutation $2n{+}1\ 2n\ 2n{-}1 \ldots 2\ 1$. □

Next, we look at sets of symmetric permutations that avoid 132 and a pattern of length four. The following lemma will be quite useful in enumerating these permutations, and we prove it using techniques from this section.

Lemma 2.22 *Fix $n \geq 0$, and R a set of permutations with $132 \in R$. Then the following are forbidden patterns in $B_n^s(R)$ and $B_n^t(R)$.*

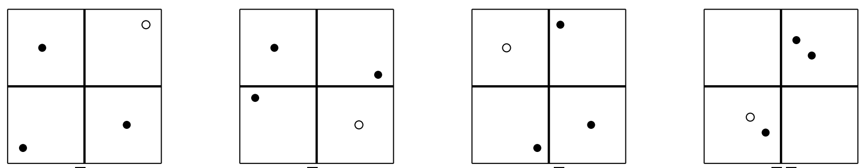
(i) $\bar{1}2$

(ii) $\bar{2}1$

(iii) $2\bar{1}$

(iv) $\bar{2}\bar{1}$

Proof. See the accompanying diagrams.



$\bar{1}2$ $\quad$ $\bar{2}1$ $\quad$ $2\bar{1}$ $\quad$ $\bar{2}\bar{1}$

□

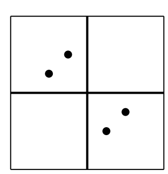
Lemma 2.23 *Fix $n \geq 0$, and suppose R is a set of patterns with $132 \in R$. If $\pi \in B_n^s(R)$ or $\pi \in B_n^t(R)$, then $bar(\pi) = [n] \setminus [k]$ for some k, with $0 \leq k \leq n$.*

Proof. Suppose this were not the case. Then there exist a and b, with $a < b < n$, where $a, n \in bar(\pi)$ and $b \notin bar(\pi)$. But Lemma 2.22(i) and (ii) imply that all the barred entries of π appear to the right of the unbarred entries, so a appears to the right of b. But then $b\bar{a}$ is of type $2\bar{1}$, which is forbidden by Lemma 2.22(iii). □

Theorem 2.24 *For all $n \geq 0$,*

$$|S_{2n}^{rc}(132, 3412)| = |S_{2n+1}^{rc}(132, 3412)| = n + 1. \tag{10}$$

Proof. We claim that for each k, $0 \leq k \leq n$, the set $[n] \setminus [k]$ is the set of bars in exactly one permutation $\pi \in B_n^s(132, 3412)$, namely $\pi = 12 \ldots k\, \overline{k+1}\, \overline{k+2} \ldots \overline{n}$. We already know from Lemma 2.23 that $bar(\pi)$ is of the from $[n] \setminus [k]$, and from Lemma 2.22 that all the barred entries are to the right of the unbarred entries and in increasing order. Similarly, all the unbarred entries must be increasing, or else we get a 3412 subsequence:



21

Thus, the form above is forced. Since each value of k, with $0 \leq k \leq n$, yields a permutation, we have $|S_{2n}^{rc}(132, 3412)| = n + 1$. It is straightforward to check that $\pi \in B_n^s(132, 3412)$ if and only if $\pi \in B_n^t(132, 3412)$, which completes the proof. □

Now, we present another similar result.

Theorem 2.25 *For all $n \geq 0$,*

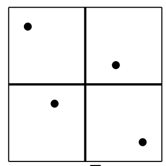
$$|S_{2n}^{rc}(132, 4321)| = |S_{2n+1}^{rc}(132, 4321)| = n + 1. \tag{11}$$

Proof. The proof of (11) is similar to the proof of (10) except all unbarred entries in $\pi \in B_n^s(132, 3412)$ (or $\pi \in B_n^t(132, 3412)$) must be descending to avoid 4321. $\square$

The last set we examine in this section is $S_n^{rc}(132, 4231)$. This set has slightly different restrictions than our last two, so we first prove the following lemma.

Lemma 2.26 *If $\pi \in B_n^s(132, 4231)$ or $\pi \in B_n^t(132, 4231)$, then $bar(\pi) = \emptyset$ or $bar(\pi) = [n]$.*

Proof. This diagram shows that $1\bar{2}$ is a forbidden subsequence of $\pi \in B_n^s(132, 4231)$ and $\pi \in B_n^t(132, 4231)$:



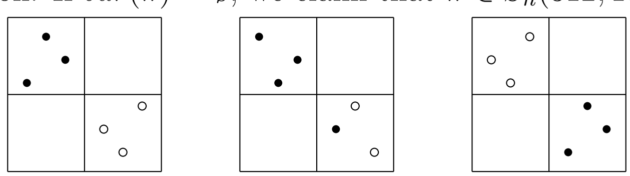
$1\bar{2}$

Suppose $\pi \in B_n^s(132, 4231)$ (or $\pi \in B_n^t(132, 4231)$) and $\pi(a)\pi(b)$ is a subsequence in which exactly one of $\pi(a)$ and $\pi(b)$ is barred. If $\pi(a)$ is barred, then $\pi(a)\pi(b)$ is either a $\bar{1}2$ or $\bar{2}1$ subsequence, which are both forbidden by Lemma 2.22. If $\pi(b)$ is barred, then $\pi(a)\pi(b)$ is a $1\bar{2}$ or $2\bar{1}$ subsequence, which are forbidden above and by Lemma 2.22, respectively. The result follows. $\square$

Theorem 2.27 *For all $n \geq 0$,*

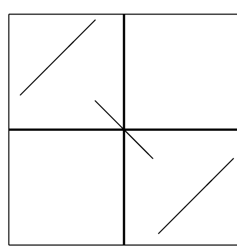
$$|S_{2n}^{rc}(132, 4231)| = |S_{2n+1}^{rc}(132, 4231)| = n + 1. \tag{12}$$

Proof. Suppose $\pi \in B_n^s(132, 4231)$. By Lemma 2.26, there are two cases to consider: $bar(\pi) = [n]$ and $bar(\pi) = \emptyset$. If $bar(\pi) = [n]$, then $\pi = \bar{1}\bar{2}\ldots\bar{n}$ by Lemma 2.22, and π^s is the identity permutation. If $bar(\pi) = \emptyset$, we claim that $\pi \in S_n(312, 132, 231)$:



312 132 231

$\pi \in S_n(312, 132, 231)$ implies that π^s is of the following form.



One can easily verify that permutations of the above form avoid 132 and 4231. Thus,

$$|S_{2n}^{rc}(132, 4231)| = 1 + |S_n(312, 132, 231)| = 1 + n,$$

where the last equality is shown by taking the reverse of the permutations in [5, Prop. 16*, $\mathcal{C}$]. One can easily verify that $\pi \in B_n^s(132, 4231)$ if and only if $\pi \in B_n^t(132, 4231)$, and the proof is complete. $\square$

3 Where Number 1?

In this section we base our proofs on asking what positions the 1 can be in and then enumerating each resulting case. Our next theorem is a straight-forward use of this case analysis.

Theorem 3.1 *For all $n \geq 0$,*

$$|S^{rc}_{2n+1}(123, 4312)| = 1, \tag{13}$$

and for all $n \geq 2$,

$$|S^{rc}_{2n}(123, 4312)| = 6. \tag{14}$$

In particular, if $\pi \in S^{rc}_{2n+1}(123, 4312)$, then $\pi = 2n\ 2n-1 \dots 2\ 1$.
Furthermore, if $\pi \in S^{rc}_{2n}(123, 4312)$, then π is one of the following.

(i) $n \dots 1\ 2n \dots n{+}1$

(ii) $n{+}1\ n{-}1 \dots 1\ 2n \dots n{+}3\ n{+}2\ n$

(iii) $n \dots 2\ 2n\ 1\ 2n{-}1 \dots n{+}1$

(iv) $n{+}1\ n-1 \dots 2\ 2n\ 1\ 2n{-}1 \dots n{+}2\ n$

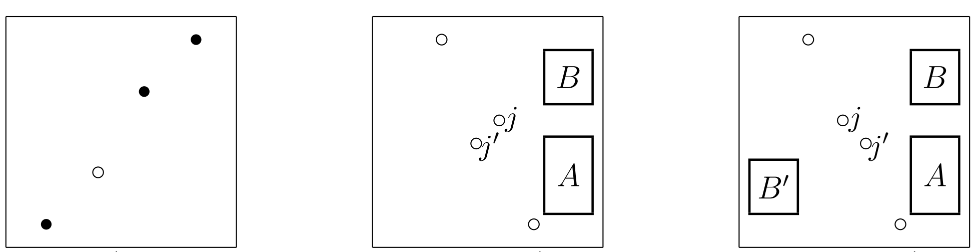
(v) $2n\ n \dots 2\ 2n{-}1 \dots n{+}1\ 1$

(vi) $2n \dots 1$

Proof. Suppose $\pi \in B^t_n(123, 4312)$. We claim that $\pi = 123 \dots n$. From Lemma 2.3, $\pi \in S_n$. Now suppose that π contains a descent $\pi(a)\pi(b)$. Then $\pi^t(a)\pi^t(b)$ is an ascent (as is their image under rc), and $\pi^t(a)\pi^t(n{+}1)\pi^t(2n{+}2{-}a)\pi^t(2n{+}2{-}b)$ is a 4312 subsequence. Thus, π is entirely increasing, and (13) follows.

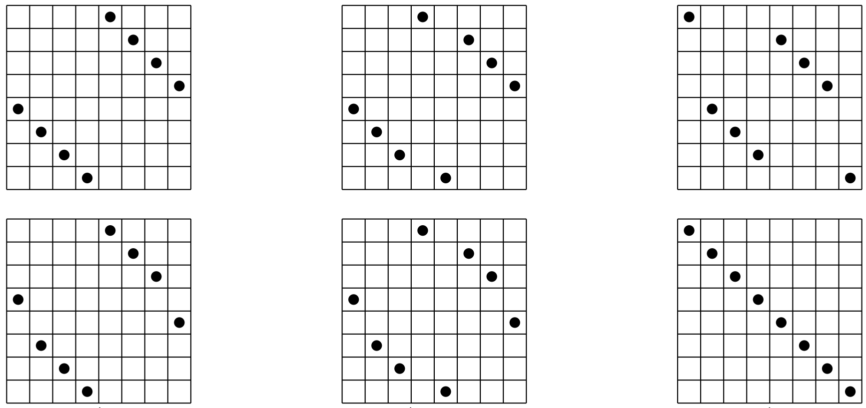
Now consider $\pi \in S^{rc}_{2n}(123, 4312)$. We claim that there are only three possible values for $\pi^{-1}(1)$ (the location of 1 in π): n, $n+1$, and $2n$. The argument appears in the diagrams below. In diagram (i), we see that if 1 is located to the left of position n, we get a 123 subsequence. In diagrams (ii) and (iii) we claim to cover all the cases when 1 is between $n+1$ and $2n$. Notice that there must be some element between $2n$ and 1, call it j, because $\pi^{-1}(1) > n{+}1$. Furthermore, there must be some element to the right of 1, call it k, because $\pi^{-1}(1) < 2n$.

Consider j along with its image under rc, call it j'. These elements can either be increasing or decreasing, as shown in the second and third diagrams, respectively. Also, consider the location of k. If $k < j$, then k is in box A. If $k > j$, then k is in box B. In both (ii) and (iii), if k is in box A, then $2n\ j\ 1\ k$ form a 4312-type subsequence. In (ii), if k is in box B, then $j'\ j\ k$ form a subsequence of type 123. Finally, in (iii), if k is in box B, then the k's image, say k' is in B', and $k'\ j\ k$ form a 123 subsequence.



(i) $\pi^{-1}(1) < n$ (ii) $n+1 < \pi^{-1}(1) < 2n$ (iii) $n+1 < \pi^{-1}(1) < 2n$

Since there are only three possible locations for the 1, we claim that the following six diagrams represent the only six diagrams in $S_{2n}(123, 4312)$ for all $n \geq 2$.



(i) $\pi^{-1}(1) = n$ (ii)$\pi^{-1}(1) = n+1$ (iii) $\pi^{-1}(1) = 2n$

In the first two cases, all elements to the left of 1 and $2n$ must be descending or else an ascent with $2n$ forms a 123 subsequence. It follows by symmetry that all elements to the right of 1 and $2n$ must be descending as well. At most one element to the left is greater than any element to the right, or else we'd have i and j to the left of 1 both greater than k to the right, but i, j in order is a decrease, so $i, j, 1, k$ is a 4312 subsequence. By symmetry this produces exactly two results in each case. In the third case, we need only require that the inner $2n-2$ length permutation avoids 123 and 312 for which there are 2 solutions by Egge [3, Thm 2.10 (iv)]. $\square$

Next, we enumerate $S_n^{rc}(123, 2431)$ by conditioning on the location of the 1.

Lemma 3.2 *If $\pi \in S_{2n}^{rc}(123, 2431)$ and $\pi^{-1}(1) = k > n+1$, then for all $i < k < j$, we have $\pi(i) > \pi(j)$.*

Proof. Suppose not, so there exist i, j such that $i < k < j$ and $\pi(i) < \pi(j)$. If $2n+1-k < i < k$, then $\pi(j), 2n, \pi(i), 1$ is a 2431 subsequence. If not, then $i < 2n+1-k$. In that case, if $\pi(i) \leq n$, then since $k > n+1$, there exists an element l such that $2n+1-k < l < k$, and by symmetry we may assume $\pi(l) \geq n+1$. Now $\pi(i), 2n, \pi(l), 1$ forms a 2431 subsequence. Finally, if $\pi(i) \geq n+1$, then since $\pi(j) > \pi(i)$, $\pi(j) > n+1$. If $2n+1-i < j$, then $1, \pi(2n+1-i), \pi(j)$ forms a 123 subsequence. Otherwise, $\pi(i), 2n, \pi(j), \pi(2n+1-i)$ forms a 2431 subsequence. This exhausts all the cases. $\square$

Lemma 3.3 *If $\pi \in S_{2n}^{rc}(123, 2431)$ and $\pi^{-1}(1) = k > n+1$, then $\pi = 2n-1\ 2n-2 \dots k\ 2n\ \sigma\ 1\ 2n+1-k \dots 3\ 2$, where $\sigma \in S_{2(k-n-1)}^{rc}(123, 132)$.*

Proof. By Lemma 3.2, the last $2n-k+1$ elements exactly comprise the set $[k]$. Furthermore, since $\pi(k) = 1$, all the elements after k must appear in decreasing order, or else there exists a 123 subsequence. By symmetry, this completes the proof up to the identity of σ. Clearly $\sigma \in S_{2(k-n-1)}^{rc}(123, 132)$, since 123 is forbidden in π, and a 132 subsequence in σ along with $\pi(k) = 1$ would produce a 2431 subsequence. This is also sufficient: π does not contain 123, since the elements before σ are all greater than σ and avoid 123, and the elements after σ are all less than σ and also avoid 123. Furthermore, 2431 is only skew decomposable into

243 and 1, so the only way a 2431 subsequence could appear is if some section of π contained 132. $\square$

Lemma 3.4 *If $\pi \in S_{2n}^{rc}(123, 2431)$ and $\pi^{-1}(1) = n$ or $n+1$, then either*

(i) *For all $i < n$ and $j > n+1$, we have $\pi(i) > \pi(j)$*

or

(ii) *For all $i < n$ and $j > n+1$, we have $\pi(i) < \pi(j)$*

Proof. Suppose not. In particular, this means that there exist $i, j < n$ such that $\pi(i) \leq n, \pi(j) \geq n+1$. If $i < j$, then $\pi(i), \pi(j), 2n$ forms a 123 subsequence. If $i > j$, then $\pi(j), 2n, \pi(2n+1-j), \pi(2n+1-i)$ forms a 2431 subsequence. $\square$

Lemma 3.5 *If $\pi \in S_{2n}^{rc}(123, 2431)$ and $\pi^{-1}(1) \leq n+1$, then π is one of the following four permutations.*

(i) $n\ n-1 \dots 2\ 1\ 2n\ 2n-1 \dots n+2\ n+1$

(ii) $2n-1\ 2n-2\ \dots n+2\ n+1\ 1\ 2n\ n\ n-1 \dots 3\ 2$

(iii) $n\ n-1 \dots 2\ 2n\ 1\ 2n-1 \dots n+2\ n+1$

(iv) $2n-1\ 2n-2 \dots n+2\ n+1\ 2n\ 1\ n\ n-2 \dots 3\ 2$

Proof. The above four all avoid 123 and 2431, and Lemma 3.4 shows that no other permutations are possible. $\square$

For the proof of the next theorem, we will require the following identities by Egge ([3, Thm. 2.10(iii)]).

$$|S_{2n}^{rc}(123, 132)| = F_{n+1} \tag{15}$$

$$|S_{2n+1}^{rc}(123, 132)| = F_n \tag{16}$$

where $F_0 = F_1 = 1, F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$.

Theorem 3.6 *For all $n \geq 1$,*

$$|S_{2n}^{rc}(123, 2431)| = F_{n+2} + 1. \tag{17}$$

Proof. We condition on $\pi^{-1}(1) = k$. If $k = n$ or $k = n+1$, then there are two possibilities by Lemma 3.5. Otherwise we get

$$\begin{aligned} \sum_{k=n+2}^{2n} |S_{2(k-n-1)}^{rc}(123, 132)| &= \sum_{k=n+2}^{2n} F_{k-n+1} \\ &= \sum_{k=3}^{n+1} F_{k-1} \\ &= F_{n+2} - 3, \end{aligned}$$

so $|S_{2n}^{rc}(123, 2431)| = 4 + F_{n+2} - 3 = F_{n+2} + 1.$ $\square$

Theorem 3.7 *For all $n \geq 0$,*

$$|S^{rc}_{2n+1}(123, 2431)| = F_{n+1} - 1 \tag{18}$$

Proof. We use an identical technique as in the proof of Theorem 3.6, except that σ in Lemma 3.3 now has length $2(k-n-1)+1$, and Lemmas 3.4 and 3.5 are irrelevant since $\pi^{-1}(1)$ must be greater than $n+1$ to avoid a 123 subsequence. This yields

$$\begin{aligned} |S^{rc}_{2n+1}(123, 2431)| &= \sum_{k=n+2}^{2n+1} |S^{rc}_{2(k-n-1)+1}(123, 132)| \\ &= \sum_{k=n+2}^{2n+1} F_{k-n-2} \\ &= \sum_{k=1}^{n} F_{k-1}, \end{aligned}$$

which is equal to $F_{n+1} - 1$ by [1, Id. 1]. □

For the rest of this section we enumerate various sets of rc-symmetric permutations which avoid 132 and a pattern of length four. The proofs of the following lemmas and theorems have a similar flavor to the earlier proofs in this section. First, we look at $S^{rc}_n(132, 3421)$, which is just like $S^{rc}_n(123, 4312)$ in the sense that there are a constant number of permutations in $S^{rc}_{2n}(132, 3421)$ and a constant number of permutations in $S^{rc}_{2n+1}(132, 3421)$ for sufficiently large n.

Theorem 3.8 *For all $n \geq 2$,*

$$|S^{rc}_{2n}(132, 3421)| = 4, \tag{19}$$

and

$$|S^{rc}_{2n+1}(132, 3421)| = 3. \tag{20}$$

In particular, the permutations in $S^{rc}_n(132, 3421)$ are of the following form, where diagrams (i)-(iii) *are in $S^{rc}_{2n+1}(132, 3421)$ and diagrams* (i)-(iv) *are in $S^{rc}_{2n}(132, 3421)$:*

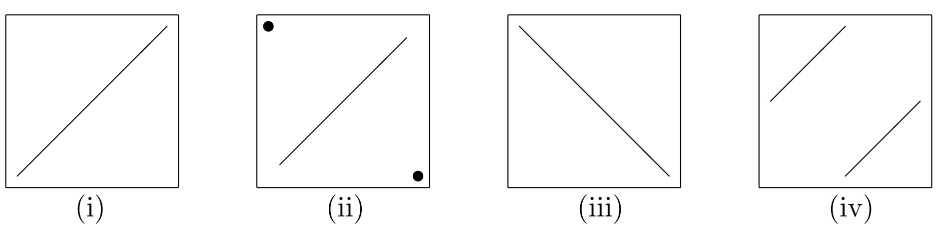
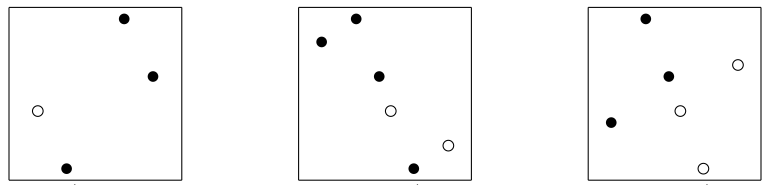


Figure 4: The possible forms of $\pi \in S^{rc}_n(132, 3421)$

Proof. Suppose $\pi \in S^{rc}_{2n}(132, 3421)$. We show that $\pi^{-1}(1)$ (the location of the 1 in π) has three possible values: 1, $n+1$, and $2n$. Suppose that $1 < \pi^{-1}(1) < n+1$. Then there is some entry j to the right of $2n$, and 1 $2n$ j is a 132-type subsequence. On the other hand, if $n+1 < \pi^{-1}(1) < 2n$, then there is some entry j between $2n$ and 1 and some entry k before $2n$. If $k > j$, then k $2n$ j 1 is a 3421 pattern. If $k < j$, then k $2n$ j is a 132 pattern. We show each of these cases graphically.



$1 < \pi^{-1}(1) < n+1$ $\quad$ $n+1 < \pi^{-1}(1) < 2n$, $k > j$ $\quad$ $n+1 < \pi^{-1}(1) < 2n$, $k < j$

To finish the proof, we must only show that (i)-(iv) are in fact the only permutations possible. First, suppose $\pi^{-1}(1) = 1$. Then everything to the right of it must be ascending to avoid 132, which forces (i). If $\pi^{-1}(1) = 2n$, then the only requirement is that the inner $2n-2$ entries avoid 132 and 231, which forces solutions (ii) and (iii), by [3, Thm 2.10 (iv)].

Finally, if $\pi^{-1}(1) = n+1$, then, as before, everything must be ascending to the right of the 1. Furthermore, everything right of the 1 must be smaller than everything left of the 1. Otherwise, suppose there was a j to the left and a k to the right with $j < k$. Then $j\ 2n\ k$ is a 132 subsequence. This forces (iv) and (19) follows.

The proof of (20) is similar, except $\pi \in S^{rc}_{2n+1}$ implies $\pi(n+1) = n+1$, which makes (iv) impossible. $\square$

Next, we look at $S^{rc}_m(132, 1234)$.

Lemma 3.9 *If $\pi \in S^{rc}_m(R)$ and $132 \in R$, then for all i, j, with $i < \pi^{-1}(1) < j$, we have $\pi(i) > \pi(j)$.*

Proof. Suppose not. Then $m+1-\pi(j)\ m\ m+1-\pi(i)$ is a 132 subsequence. $\square$

Lemma 3.10 *If $\pi \in S^{rc}_m(R)$, $132 \in R$, and $\pi^{-1}(1) < \pi^{-1}(m)$, then π is $1\ 2\ \ldots\ m$.*

Proof. All entries following 1 in π must be in ascending order, or else we have a 132 subsequence. Since m follows 1, m must be the last entry of π, so $\pi^{-1}(1) = 1$, and it follows that π is $1\ 2\ \ldots\ m$. $\square$

Lemma 3.11 *If $\pi \in S^{rc}_m(132, 1234)$, then $\pi^{-1}(1) \geq m-2$.*

Proof. Suppose not. Then there are at least three entries of π which follow 1. If any of these are in descending order, say $\pi^{-1}(1) < i < j$ and $\pi(i) > \pi(j)$, then $1\pi(i)\pi(j)$ is a 132 subsequence. If they are all ascending, say $\pi^{-1}(1) < i < j < k$ and $\pi(i) < \pi(j) < \pi(k)$, then $1\pi(i)\pi(j)\pi(k)$ is a 1234 subsequence. $\square$

Lemma 3.12 *If $\pi \in S^{rc}_m(132, 1234)$, then π has one of the following three forms.*

(i) $\pi = 1 \ominus \sigma \ominus 1$, *where* $\sigma \in S^{rc}_{m-2}(132, 1234)$.

(ii) $\pi = 12 \ominus \sigma \ominus 12$, *where* $\sigma \in S^{rc}_{m-4}(132, 1234)$.

(iii) $\pi = 123 \ominus \sigma \ominus 123$, *where* $\sigma \in S^{rc}_{m-6}(132, 1234)$.

Proof. The possible locations of the 1 are given by Lemma 3.11. The entries following it are given by Lemma 3.9, and must be in increasing order to avoid a 132 subsequence. The rest is determined by symmetry, and the fact that if σ contained a 132 or 1234 subsequence, so would π. $\square$

Theorem 3.13 *For all $n \geq 0$,*

$$|S_{2n}^{rc}(132, 1234)| = |S_{2n+1}^{rc}(132, 1234)| = T_n, \tag{21}$$

where T_n is the nth Tribonacci number given by the recurrence $T_0 = 1, T_1 = 2, T_2 = 3, T_{n+1} = T_n + T_{n-1} + T_{n-2}$ for $n \geq 2$.

Proof. It is easy enough to confirm this theorem up to $n = 2$, since there are at most 8 permutations to be checked for any given n. The recurrence is given by Lemma 3.12, after noting that any σ avoiding 132 and 1234 will suffice: since any such π is skew decomposable (by definition), and 132 and 1234 are *not* skew decomposable, any 132 or 1234 subsequence of π must be contained in a single summand. Since neither 123 nor σ contains such a pattern, neither does π. $\square$

Now we examine $S_m^{rc}(132, 2341)$. We introduce the following lemmas regarding the structure of permutations in this set.

Lemma 3.14 *If $\pi \in S_m^{rc}(132, 2341)$ and $\pi^{-1}(1) > \pi^{-1}(m)$, then $\pi^{-1}(1) \geq m - 1$.*

Proof. Suppose not. Then there are at least two entries of π which follow 1, say $\pi^{-1}(1) < i < j$. If $\pi(i) > \pi(j)$, then $1\pi(i)\pi(j)$ is a 132 subsequence. If $\pi(i) < \pi(j)$, then $m + 1 - \pi(j), m + 1 - \pi(i), m, 1$ is a 2341 subsequence. $\square$

Lemma 3.15 *If $\pi \in S_m^{rc}(132, 2341)$, then π has one of the following three forms:*

(i) $\pi = 1 \ominus \sigma \ominus 1$, *where $\sigma \in S_{m-2}^{rc}(132, 123)$.*

(ii) $\pi = 12 \ominus \sigma \ominus 12$, *where $\sigma \in S_{m-4}^{rc}(132, 123)$.*

(iii) $\pi = 123 \ldots m$.

Proof. The first two forms follow from Lemmas 3.14 and 3.9. σ must avoid 123, or else the 123 sequence adjoined with the 1 in π creates a 2341 subsequence in π. The last form is the only possibility if $\pi^{-1}(1) < \pi^{-1}(m)$, as follows from Lemma 3.10. $\square$

With these lemmas in hand, we are ready to enumerate $S_m^{rc}(132, 2341)$.

Theorem 3.16 *For all $n \geq 2$,*

$$|S_{2n}^{rc}(132, 1234)| = F_{n+1} + 1. \tag{22}$$

Proof. We claim that if a permutation π is of one of the forms given in Lemma 3.15, then $\pi \in S_m^{rc}(132, 2341)$. Clearly (iii) is. For (i) and (ii), note that since 132 is not skew decomposable, any 132 subsequence must appear in a single summand of π. Likewise, since 2341 is only skew decomposable into 123 and 1, a 2341 subsequence can only appear in π if 123 appears in a summand of π. Combining (15) with Lemma 3.15, we find $|S_{2n}^{rc}(132, 1234)| = |S_{2n-2}^{rc}(132, 123)| + |S_{2n-4}^{rc}(132, 123)| + 1 = F_n + F_{n-1} + 1 = F_{n+1} + 1$. $\square$

Theorem 3.17 *For all $n \geq 1$,*

$$|S_{2n+1}^{rc}(132, 2341)| = F_n + 1. \tag{23}$$

Proof. By the same argument as in the proof of Theorem 3.16, any permutation of the forms in Lemma 3.15 is valid. Using (16) from earlier in the paper, we find that $|S^{rc}_{2n+1}(132, 2341)| = |S^{rc}_{2n-1}(132, 123)| + |S^{rc}_{2n-3}(132, 123)| + 1 = F_{n-1} + F_{n-2} + 1 = F_n + 1$. □

Theorem 3.17 completes our analysis of pattern-avoiding permutations avoiding one pattern of length three and one pattern of length four. We note that of the $3! \cdot 4! = 144$ possible pairs $\sigma \in S_3, \tau \in S_4$, only the twelve pairs in the table below are of interest. For instance, there is no need to enumerate $S^{rc}_n(123, 2314)$ because 2314 contains the pattern 123, so $|S^{rc}_n(123, 2314)| = |S^{rc}_n(123)|$, which has been enumerated by Egge [3]. Similarly, we have Wilf-equivalence classes such as $|S^{rc}_n(123, 1432)| = |S^{rc}_n(321, 2341)| = |S^{rc}_n(321, 4123)|$, achieved by reversing the entries and taking the complements of entries, respectively. With the exception of $S^{rc}_{2n}(123, 1432)$, our results are complete.

σ	τ	$\lvert S^{rc}_{2n}(\sigma,\tau)\rvert$	$\lvert S^{rc}_{2n+1}(\sigma,\tau)\rvert$
123	2413	F_{2n+1}	F_{2n-1}
123	2431	$F_{n+3}+1$	$F_{n+2}-1$
123	3412	$2^{n+1}-(n+1)$	1
123	4231	n^2+1	$\binom{n}{2}+1$
123	4312	6	1
123	1432	?	A116716
132	1234	T_n	T_n
132	2341	$F_{n+1}+1$	F_n+1
132	3412	$n+1$	$n+1$
132	4231	$n+1$	$n+1$
132	4321	$n+1$	$n+1$
132	3421	4	3

A Appendix

Here we prove the Fibonacci identity we used in the proof of Theorem 2.8. We also give a natural generalization of the identity. Neither this identity nor its generalization appears in [1].

Theorem A.1 *For all $n \geq 1$,*

$$F_{2n-2} + \sum_{k=1}^{n} F_{2k-4} 2^{n-k} = F_{2n} \tag{24}$$

Proof. This is equivalent to showing $\sum_{k=1}^{n} F_{2k-4}2^{n-k} = F_{2n-1}$. We prove this identity using techniques of [1]. Accordingly, we recall that F_n is the number of ways to tile a $1 \times n$ board using 1×1 square tiles and 1×2 domino tiles.

How many ways can we tile a board of length $2n - 1$?

Answer 1: F_{2n-1}

Answer 2: In the diagram below, consider the *odd fault lines* indicated by dotted lines. A fault line is called *unbreakable* if a domino lies across it. Suppose the right-most unbreakable odd fault line is at index $2k - 3$. Then a domino lies across the fault and a square must lie directly to its right (these are shaded in the figure below) or else the next fault-line would be unbreakable as well.

Now, there are $2(n-k)$ cells to the right that need to be filled, but each fault-line must be breakable, so we can fill each pair of cells with either two squares or a domino. Thus, there are 2^{n-k} ways to fill the right side of the board. The $2k-4$ cells to the left can be freely tiled in F_{2k-4} ways. Note that when $k=1$, there is no unbreakable fault line, which means there are no cells to the left that are unaccounted for. Thus, we are tiling an empty board, and we say in this case that $F_{-2}=1$ (a somewhat curious claim, but a natural extension of the Fibonacci numbers to negative indices). Summing over k, $1 \leq k \leq n$, gives us (24).

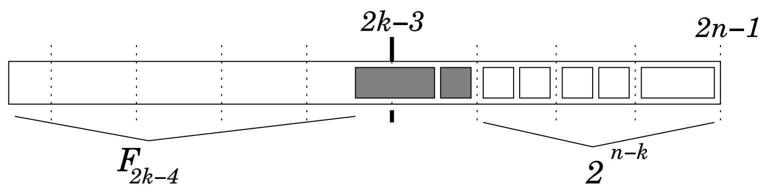


Figure 5: Conditioning on the rightmost unbroken odd fault line

□

Comment. A natural generalization of the above identity is

$$F_{mn+r} = F_r F_m^n + \sum_{k=1}^{n} F_{mk-m+r-1} F_{m-1} F_m^{n-k} \tag{25}$$

We leave the proof of (25), which is similar to the proof of (24), as an exercise for the reader.

B Acknowledgments

Funding for this research was provided by the Carleton College Howard Hughes Medical Institute grant. We would like to thank our advisor, Eric Egge, for his guidance, patience, and helpful comments.

References

[1] A. Benjamin, J. Quinn. *Proofs that Really Count.* The Mathematical Association of America, 2003.

[2] E. S. Egge. Restricted 3412-avoiding involutions, continued fractions, and Chebyshev polynomials. *Adv. Appl. Math.*, 33:451–475, 2004.

[3] E. S. Egge. Restricted Symmetric Permutations. *Ann. Comb.*, 11: 405-434, 2007.

[4] T. Mansour, S. H. F. Yan, and L. L. M. Yang. Counting occurrences of 231 in an involution. *Discrete Math.*, 306(6):564–572, 2006.

[5] R. Simion and F. Schmidt. Restricted permutations. *Europ. J. Combin.*, 6:383–406, 1985.

[6] The On-Line Encyclopedia of Integer Sequences, http://www.research.att.com/njas/sequences/.

[7] J. West. Generating trees and forbidden subsequences. *Discrete Math.*, 157:363–374, 1996.