# Identities for Triangular Recurrences

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## Convolutions of Triangular Recurrences

$$\sum_{k=m}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \binom{k}{m} = \begin{bmatrix} n+1 \\ m+1 \end{bmatrix}$$

This equation is known to be true for the Stirling numbers of the 1st kind. A similar equation holds for the Stirling numbers of the 2nd kind. So, how can we adapt them to fit the Legendre-Stirling numbers?

We define a function of n and m to equal the above sum. Then, we can draw up a new triangle for that function.

$$\sum_{k=m}^{n} \left[\begin{bmatrix} n \\ k \end{bmatrix}\right] \binom{k}{m} = \left(\begin{bmatrix} n \\ m \end{bmatrix}\right)$$

**Triangle for $\left(\begin{bmatrix} n \\ m \end{bmatrix}\right)$**

n=0   1

n=1   1   1

n=2   3   4   1

…..   21   31   11   1

273   424   174   24   1

This new function also satisfies a recurrence

$$\left(\begin{bmatrix} n \\ m \end{bmatrix}\right) = \left(\begin{bmatrix} n-1 \\ m-1 \end{bmatrix}\right) + \big((n)(n-1)+1\big)\left(\begin{bmatrix} n-1 \\ m \end{bmatrix}\right)$$

For comparison, here is the recurrence for the Legendre-Stirling numbers of the 1st kind, which is remarkably similar.

$$\left[\begin{bmatrix} n \\ k \end{bmatrix}\right] = \left[\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\right] + (n)(n-1)\left[\begin{bmatrix} n-1 \\ k \end{bmatrix}\right]$$

Before going any further, it's important to introduce some new notation.

$$\binom{n}{k}_{a,b} = a\binom{n-1}{k-1}_{a,b} + b\binom{n-1}{k}_{a,b}$$

Using this notation, I can write out a generalized formula for two generic triangular recurrences.

$$\sum_{k=m}^{n} \binom{n}{k}_{a,b} \binom{k}{m}_{c,d} = \binom{n}{m}_{ac,b+ad}$$

Note that a, b can be functions of n, but not of k. Also, c and d can be functions of k, not of n. For example:

$$\sum_{k=m}^{n} \binom{n}{k} \left\{\begin{Bmatrix} k \\ m \end{Bmatrix}\right\} = \binom{n}{m}_{1,k(k+1)+1}$$

$$\sum_{k=m}^{n} \left[\begin{bmatrix} n \\ k \end{bmatrix}\right] \left\{\begin{Bmatrix} k \\ m \end{Bmatrix}\right\} = \binom{n}{m}_{1,k(k+1)+n(n-1)}$$

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## Pascal's Triangle

1

1   1

1   2   1

1   3   3   1

1   4   6   4   1

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$\binom{n}{k}$ = the number of ways to choose k elements from n elements, often read as "n choose k"

## Stirling Numbers of the First Kind

1

1   1

2   3   1

6   11   6   1

24   50   35   10   1

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}$$

$\begin{bmatrix} n \\ k \end{bmatrix}$ = the number of ways to put n elements into k distinct cycles, often read "n cycle k"

## Stirling Numbers of the Second Kind

1

1   1

1   3   1

1   7   6   1

1   15   25   10   1

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} + k\begin{Bmatrix} n-1 \\ k \end{Bmatrix}$$

$\begin{Bmatrix} n \\ k \end{Bmatrix}$ = the number of ways to break up n elements into k distinct partitions, often read as "n partition k"

## Legendre-Stirling Numbers of the First Kind

1

2   1

12   8   1

144   108   20   1

2880   2304   508   40   1

$$\left[\begin{bmatrix} n \\ k \end{bmatrix}\right] = \left[\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\right] + (n)(n-1)\left[\begin{bmatrix} n-1 \\ k \end{bmatrix}\right]$$

Legendre-Stirling numbers of the 1st kind count the number of ways to choose the following:

1. You have two permutations, $P_1$ and $P_2$.
2. $P_1$ is a permuation of {1,2,…,n}, while $P_2$ is a permutation of {1,2,…,n,n+1}
3. $P_1$ contains k cycles. $P_2$ contains k+1 cycles.
4. The largest element of each cycle in $P_1$ must also be the largest element of a cycle in $P_2$.

For example, the following is one such arrangement for n=4, k=2:

$$P_1 = [1,3][2,4]$$
$$P_2 = [3][1,4][1,5]$$

## Legendre-Stirling Numbers of the Second Kind

1

2   1

4   8   1

8   52   20   1

16   320   292   40   1

$$\left\{\begin{Bmatrix} n \\ k \end{Bmatrix}\right\} = \left\{\begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}\right\} + (k)(k+1)\left\{\begin{Bmatrix} n-1 \\ k \end{Bmatrix}\right\}$$

Legendre-Stirling numbers of the 2nd kind count the number of ways to choose the following:

1. You have two copies of each of the elements {1,2,…,n}
2. These 2n elements are split up into k different blocks, plus one "zero-block".
3. Each non-zero block must contain both copies of its minimum element.
4. Each non-zero block cannot contain more than one copy of any element other than its minimum element.
5. The "zero-block" may not contain both copies of any of its elements.
6. Only the zero-block may be empty.

For example, the following is one such arrangement for n=4, k=2 :

$$\{2_2, 3_1\}, \{1_1, 1_2, 2_1, 3_2\}, \{4_1, 4_2\}$$

## Hockey Stick Theorem

$$\sum_{k=0}^{m} \binom{n+k}{k} = \binom{n+m+1}{m}$$

The above formula is known as the hockey-stick theorem. The triangle below illustrates a proof of this theorem:

1

1   1

1   2   1

1   3   3   1

1   4   6   4   1

1   5   10   10   5   1

Induction step of proof

There is also an analog of this theorem for the Stirling numbers:

$$\sum_{k=0}^{m} (n+k)\begin{bmatrix} n+k \\ k \end{bmatrix} = \begin{bmatrix} n+m+1 \\ m \end{bmatrix}$$

If we rewrite the recurrences for Pascal's triangle and the Stirling numbers for n+k+1 instead of n, we see that the expression inside each summation is the right-hand side of the recurrence.

$$\begin{bmatrix} n+k+1 \\ k \end{bmatrix} = \begin{bmatrix} n+k \\ k-1 \end{bmatrix} + (n+k)\begin{bmatrix} n+k \\ k \end{bmatrix}$$

$$\binom{n+k+1}{k} = \binom{n+k}{k-1} + \binom{n+k}{k}$$

From this, we show that:

$$\sum_{k=0}^{m} (n+k)(n+k+1)\left[\begin{bmatrix} n+k \\ k \end{bmatrix}\right] = \left[\begin{bmatrix} n+m+1 \\ m \end{bmatrix}\right]$$

So the next step is to generalize the theorem to apply to all triangular recurrences of the form:

$$\binom{n+k+1}{k}_{1,b} = \binom{n+k}{k-1}_{1,b} + b\binom{n+k}{k}_{1,b}$$

And we can, using induction. The base case is n=1, m=1:

$$\sum_{k=0}^{1} b\binom{n+k}{k}_{1,b} = b\binom{n}{0}_{1,b} + b\binom{n+1}{1}_{1,b} =$$

$$= \binom{n+1}{0}_{1,b} + b\binom{n+1}{1} = \binom{n+2}{1}_{1,b}$$

Induction Step:

$$\sum_{k=0}^{m+1} b\binom{n+k}{k}_{1,b} =$$

$$\sum_{k=0}^{m} b\binom{n+k}{k}_{1,b} + b\binom{n+m+1}{m+1}_{1,b} =$$

$$\binom{n+m+1}{m}_{1,b} + b\binom{n+m+1}{m+1}_{1,b} = \binom{n+m+2}{m+1}_{1,b}$$

When generalized even further, we get:

$$\sum_{k=0}^{m} a^{m-k} b\binom{n+k}{k}_{a,b} = \binom{n+m+1}{m}_{a,b}$$