New Pattern-Avoiding Permutations Counted by the Schröder Numbers

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Permutations and Catalan Numbers

Theorem

If $\sigma \in S_3$ *and* $n \geq 0$ *then*

$$|S_n(\sigma)| = C_n.$$

Permutations and Catalan Numbers

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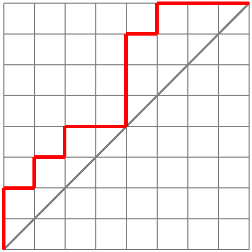
If $\sigma \in S_3$ and $n \geq 0$ then

$$|S_n(\sigma)| = C_n.$$

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

is the number of Catalan paths of length n.

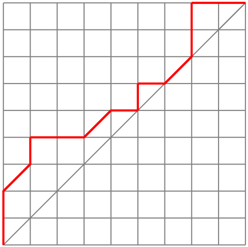
$$C_n = \sum_{j=1}^{n} C_{j-1} C_{n-j}$$



Schröder Numbers

r_n is the number of paths from $(0,0)$ to (n,n)

- using North $(0,1)$, East $(1,0)$ and Diagonal $(1,1)$ steps and
- never passing below $y = x$.



A Schröder Path

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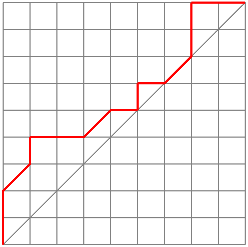

A Schröder Path

n	0	1	2	3	4	5	6	7	8
r_n	1	2	6	22	90	394	1806	8558	41586

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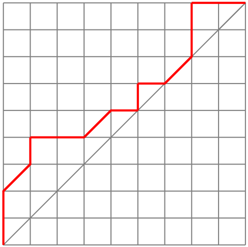

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$$r_n = r_{n-1} + \sum_{j=1}^{n} r_{j-1} r_{n-j}$$

$$r_n = \sum_{d=0}^{n} \binom{2n-d}{d} C_{n-d}$$

$|S_n(1243, 2143)| = r_{n-1}$

Observation (E and Mansour)

If $\pi \in S_n(1243, 2143)$ *then at most one entry left of* n *is smaller than some entry to the right of* n*.*

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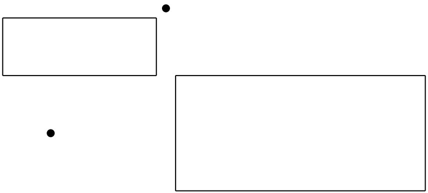

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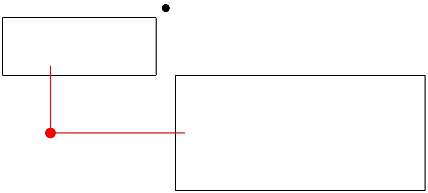

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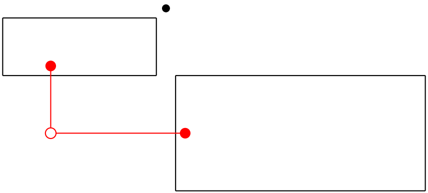

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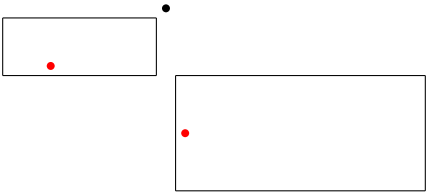

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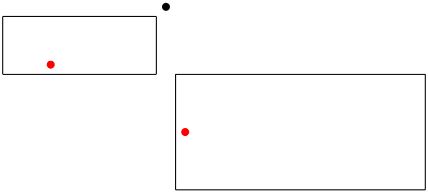

$$41523 \oplus 34251 = 8\mathbf{3}967 \quad 10 \quad 4251$$

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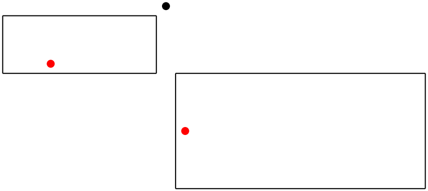

$$|S_n(,)| = |S_{n-1}(,)| + \sum_{j=1}^{n} |S_{j-1}(,)||S_{n-j}(,)|$$

Stankova's Proof that $|S_n(2413, 3142)| = r_{n-1}$

Definition

For $\pi \in S_n(2413, 3142)$,

- $L_k > L_{k-1} > \cdots > L_1$ are the maximal sets of consecutive numbers left of n.
- $R_1 > R_2 > \cdots > R_l$ are the maximal sets of consecutive numbers right of n.

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Observation

$$L_k > R_1 > L_{k-1} > R_2 > \cdots$$

or

$$R_1 > L_k > R_2 > L_{k-1} > \cdots$$

Stankova's Proof Continues

Lemma

- *L_j is to the right of L_{j-1} for all j (3142).*
- *R_j is to the right of R_{j-1} for all j (2413).*

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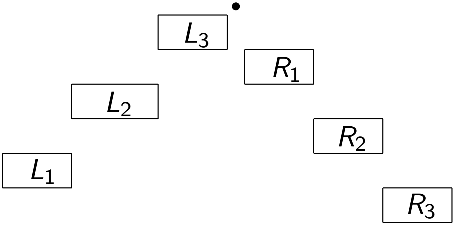

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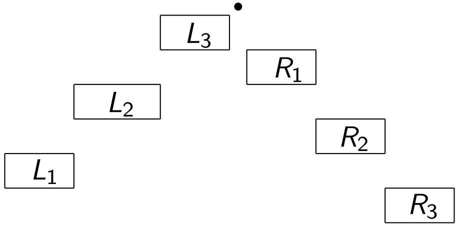

Lemma

If R_j and L_j all avoid 2413 and 3142 then so does π.

Conclusion to Stankova's Proof

If $X_n = |S_n(2413, 3142)|$ then

$$X_n = 2 \sum_{\alpha_1 + \cdots + \alpha_m = n-1} X_{\alpha_1} \cdots X_{\alpha_m},$$

which can be rewritten

$$X_n = X_{n-1} + \sum_{j=1}^{n} X_{j-1} X_{n-j}.$$

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Are there other sets R such that $|S_n(R)| = r_{n-1}$?

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Theorem (Gire,Kremer,West)

If $R \subseteq S_4$ then $|R| = 2$ and R is trivially equivalent to exactly one of

- *1234,1243*
- *1324,2314*
- *1342,2341*
- *3124,3214*
- *3142,3214*
- *3412,3421*
- *1324,2134*
- *3124,2314*
- *2134,3124*
- *2413,3142*

Avoiding Two Patterns of Length Four

Table 1 (*continued*)

Γ	$\lvert S_n(\Gamma)\rvert$, $n = 5, 6, 7, 8, 9, 10, 11$	Reference
3124,3214	90,394,1806 ,8558,41586,206098,1037718	Kremer [8]
3142,3214		
3412,3421		West [8]
1324,2134		
3124,2314		
2134,3124		
2143,2413	90,395,1823,8741,43193,218704,1129944	Open
1234,1324	90,396,1837,8864,44074,224352,1163724	Zeilberger[a]

[a]Via the package WILF (see http://www.math.temple.edu/~zeilberger).

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Question

Does there exist $\sigma \in S_6$ *such that* $|S_n(2143, 2413, \sigma)| = r_{n-1}$*?*

The Conjecture

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If σ is any of the permutations below, then $|S_n(2143, 2413, \sigma)| = r_{n-1}$.

- *415263*
- *513642*
- *624315*
- *465213*
- *516324*

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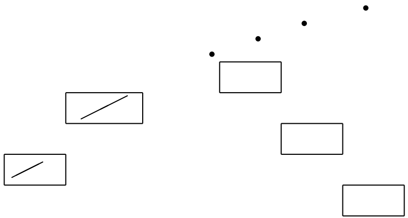
Lemma

If $\pi \in S_n(2143, 2413)$ then π has the form below.



The End

Thank You!