Introducing 05A06: Patterns in Permutations and Words

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Carleton College

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There's room for all, from undergraduates to wily veterans.

The Definition

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Suppose π and σ are permutations, written in one-line notation. An *occurrence* of σ in π is a subsequence of π whose entries are in the same relative order as the entries of σ.

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Example

3561274 contains 9 occurrences of 21.

(inversions)

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Example

3561274 contains 12 occurrences of 12.
(coinversions)

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Suppose π and σ are permutations, written in one-line notation. An *occurrence* of σ in π is a subsequence of π whose entries are in the same relative order as the entries of σ.

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3561274 contains 7 occurrences of 312.

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3561274 contains 7 occurrences of 312.

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The Definition in Pictures

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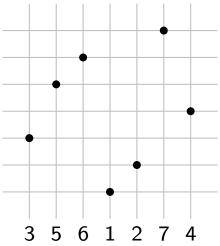

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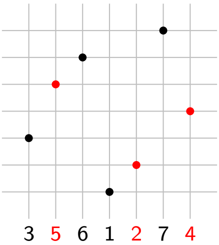

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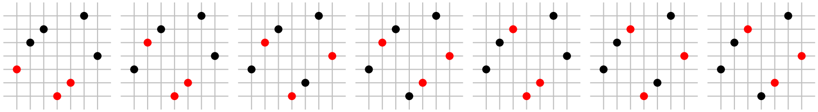

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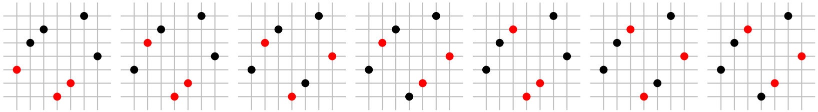

Observation

Every symmetry f of the square is a bijection between occurrences of σ in π and occurrences of σ^f in π^f.

Enumeration Questions

$\sigma[\pi] :=$ number of occurrences of σ in π

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Theorem (Rodrigues, 1839)

$$\sum_{\pi \in S_n} q^{21[\pi]} = 1(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$$

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Problem

For each σ, find $\sum_{\pi \in S_n} q^{\sigma[\pi]}$.

Enumeration Questions

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Ambition

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Opium-Induced Fever Dream

For each σ, find $\displaystyle\sum_{\pi \in S_n} q^{\sigma[\pi]}$.

Pattern Avoidance: Reining in Our Ambitions

Definition

We say π *avoids* σ whenever $\sigma[\pi] = 0$.

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Question

For each n and each σ, what is $|Av_n(\sigma)|$?

Pattern Avoidance: Reining in Our Ambitions

Definition

We say π *avoids* σ whenever $\sigma[\pi] = 0$.

$Av_n(R) = S_n(R) :=$ set of permutations in S_n which avoid all $\sigma \in R$

Question

For each n and each R, what is $|Av_n(R)|$?

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Definition

We say patterns σ_1 and σ_2 are *Wilf-equivalent* whenever

$$|Av_n(\sigma_1)| = |Av_n(\sigma_2)|$$

for all n.

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Question

Which patterns of each length are Wilf-equivalent?

Enumerative Results

σ	$\lvert Av_n(\sigma)\rvert$	OGF
123 132	$C_n = \frac{1}{n+1}\binom{2n}{n}$	$\frac{1-\sqrt{1-4x}}{2x}$

More Enumerative Results

R	$\lvert Av_n(R)\rvert$	OGF
123, 132	2^{n-1}	$\dfrac{1-x}{1-2x}$
123, 231	$1+\dbinom{n}{2}$	$\dfrac{1-2x+2x^2}{(1-x)^3}$
123, 321	0 for $n \geq 5$	$1+x+2x^2+4x^3+4x^4$
123, 132, 213	F_{n+1}	$\dfrac{1}{1-x-x^2}$
123, 132, 231	n	$\dfrac{1}{(1-x)^2}$

Even More Enumerative Results

R	$\lvert Av_n(R)\rvert$	OGF
123, 3412	$2^{n+1} - \binom{n+1}{3} - 2n - 1$	$\dfrac{1-5x+10x^2-9x^3+4x^4}{(1-2x)(1-x)^4}$
132, 4231	$1+(n-1)2^{n-2}$	$\dfrac{1-4x+5x^2-x^3}{(1-2x)^2(1-x)}$
123, 2143 123, 2413 132, 2314 132, 2341 312, 2314 312, 3241 312, 3214 123, 3214 312, 4321 312, 3421 132, 3241 132, 3412 312, 1432 312, 1342	F_{2n}	$\dfrac{1-2x}{1-3x+x^2}$

Still More Enumerative Results

R	$\lvert Av_n(R)\rvert$	OGF
2143, 3412	$\binom{2n}{n} - \sum_{m=0}^{n-1} 2^{n-m-1}\binom{2m}{m}$	$\frac{1-3x}{(1-2x)\sqrt{1-4x}}$
1234, 3214 4123, 3214 2341, 2143 1234, 2143	$\frac{4^{n-1}+2}{3}$	$\frac{x(1-3x)}{(1-x)(1-4x)}$
1324, 2143 1342, 2431 1342, 3241 1342, 2314 1324, 2413		$\frac{1-5x+3x^2+x^2\sqrt{1-4x}}{1-6x+8x^2-4x^3}$
2413, 3142 1234, 2134 1324, 2314 3124, 3214 3142, 3214 3412, 3421 1324, 2134 3124, 2314 2134, 3124	$r_{n-1} = \sum_{d=0}^{n} C_{n-d}\binom{2n-d}{d}$	$\frac{1-x-\sqrt{1-6x+x^2}}{2x}$

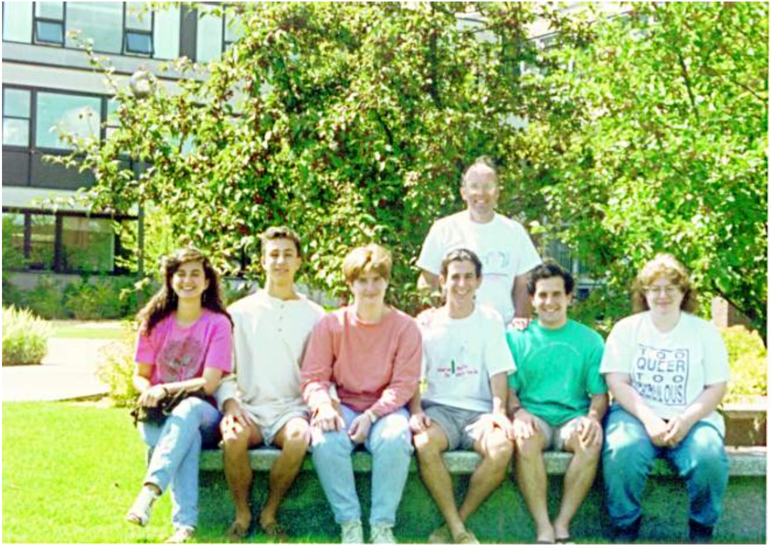
Some Open Enumerative Problems

R	$\lvert Av_n(R)\rvert$ for $n = 5, 6, 7, 8, 9, 10$
1234, 3412	86, 333, 1235, 4339, 14443, 45770
1243, 4231	86, 335, 1266, 4598, 16016, 53579
1324, 3412	86, 335, 1271, 4680, 16766, 58656
1324, 4231	86, 336, 1282, 4758, 17234, 61242
1243, 3412	86, 337, 1295, 4854, 17760, 63594
1324, 2341	87, 352, 1428, 5768, 23156, 92416
1342, 4123	87, 352, 1434, 5861, 24019, 98677
1243, 2134	87, 354, 1459, 6056, 25252, 105632
1243, 2431	88, 363, 1507, 6241, 25721, 105485
1324, 2431	88, 363, 1508, 6255, 25842, 106327
1243, 2341	88, 365, 1540, 6568, 28269, 122752
1342, 3412	88, 366, 1556, 6720, 29396, 129996
1243, 2413	88, 367, 1568, 6810, 29943, 132958
1243, 3124	88, 367, 1571, 6861, 30468, 137229
1234, 2341	89, 376, 1611, 6901, 29375, 123996
1342, 2413	89, 379, 1664, 7460, 33977, 156727
1324, 1432	89, 380, 1677, 7566, 34676, 160808
1234, 1342	89, 380, 1678, 7584, 34875, 162560
1432, 2143	89, 381, 1696, 7781, 36572, 175277
1243, 1432	89, 382, 1711, 7922, 37663, 182936
2143, 2413	90, 395, 1823, 8741, 43193, 218704

Just A Couple More Enumerative Results

σ	$\lvert Av_n(\sigma)\rvert$
1234 1243 2143 3214	$\dfrac{1}{(n+1)^2(n+2)}\displaystyle\sum_{j=0}^{n}\binom{2j}{j}\binom{n+1}{j+1}\binom{n+2}{j+1}$
1342 2413	$(-1)^{n-1}\dfrac{7n^2-3n-2}{2}+3\displaystyle\sum_{j=2}^{n}\dfrac{(2j-4)!}{j!(j-2)!}\binom{n-j+2}{2}(-1)^{n-j}2^{j+1}$
1324	Unknown beyond $n=36$

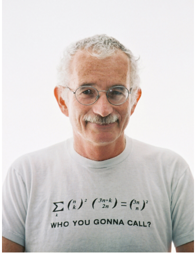
A Cool Picture



A Cool Picture

The Bet

"Not even God knows $|Av_{1000}(1324)|$."
Doron Zeilberger



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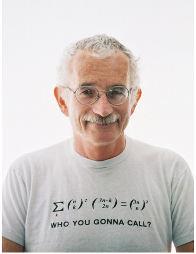

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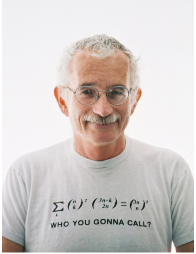

"I'm not sure how good Zeilberger's God is at math, but I believe that some humans will find this number in the not so distant future."

Einar Steingrímsson

The Stanley-Wilf Conjecture

Theorem

For all $\sigma \in S_3$,

$$\lim_{n\to\infty} \sqrt[n]{|Av_n(\sigma)|} = 4.$$

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Wilf's First Question, $\sim$ 1980

Is

$$|Av_n(\sigma)| \leq (|\sigma| + 1)^n$$

for all n?

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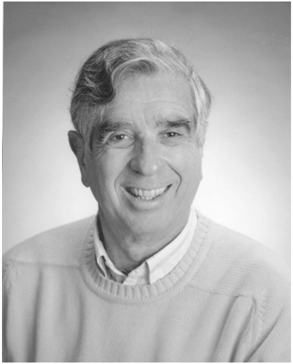

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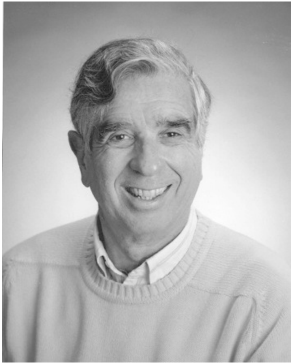

Theorem (Regev, 1981)

$$\lim_{n\to\infty} \sqrt[n]{|Av_n(12\cdots k)|} = (k-1)^2$$

The Stanley-Wilf Conjecture

Stanley's Question, $\sim$ 1980

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Wilf's Next Question

Does there exist, for each σ, a constant $c(\sigma)$ with

$$\lim_{n\to\infty} \sqrt[n]{|Av_n(\sigma)|} = c(\sigma)?$$

The Stanley-Wilf Upper Bound Conjecture

For every σ there is a constant $c(\sigma)$ such that

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Limit $\Rightarrow$ Upper Bound: Clear

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Upper Bound $\Rightarrow$ Limit: Arratia 1999

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Generalized $=$ Consecutive $=$ Vincular

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$2-41-3$

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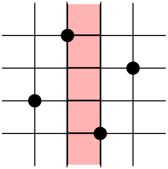
$2\underline{41}3$

$2 - 41 - 3$



Interlude: Other Notions of Containment

Generalized = Consecutive = Vincular

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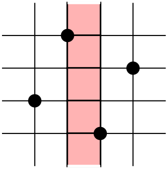


Example

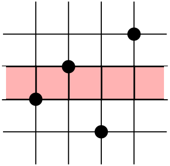
25314 contains 2413 but avoids $2\underline{41}3$.

Bivincular

Interlude: Other Notions of Containment

Bivincular

$\overline{23}14$



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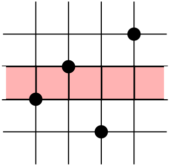

Example

315246 contains 2314 but avoids $\overline{23}14$.

The Füredi-Hajnal Conjecture

Convention: Matrices use only entries 0 and 1.

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Definition

A matrix M *contains* a matrix C whenever M has a submatrix M_{sub} of C's dimensions such that M_{sub} has a 1 in every place C has a 1.

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Example

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \text{ contains } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The Füredi-Hajnal Conjecture

The Füredi-Hajnal Question, 1992

Given a matrix C, how many 1s can an $n \times n$ matrix M contain before it must contain C?

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Theorem (Klazar, 2001)

Füredi-Hajnal $\Rightarrow$ *Stanley-Wilf*

The Marcus-Tardos Theorem

 

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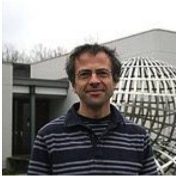
Late 2003	Marcus and Tardos prove the Füredi-Hajnal conjecture

The Marcus-Tardos Theorem



Fall 2003	Adam Marcus starts his Fulbright in Hungary, working with Gábor Tardos
Late 2003	Marcus and Tardos prove the Füredi-Hajnal conjecture
Weeks Later	Marcus and Tardos learn about the Stanley-Wilf conjecture

How Long Did It Take to Prove the Stanley-Wilf Conjecture?

Richard Stanley before

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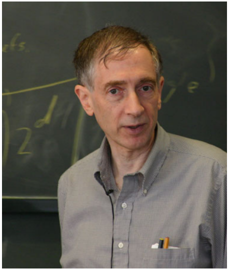

Richard Stanley after

Growth Rates

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For each σ,

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σ	$L(\sigma)$
123 132	4
1234 1243 2143 3214	9
1342 2413	8
1324	
$12\cdots k$	$(k-1)^2$

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Theorem (Bevan, 2014)

$$L(1324) \geq 9.81$$

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Theorem (Bóna, 2013)

$$L(1324) \leq 13.738$$

σ	$L(\sigma)$
123 132	4
1234 1243 2143 3214	9
1342 2413	8
1324	[9.81,13.738]
$12\cdots k$	$(k-1)^2$

Conjecture (Claesson, Jelínek, Steingrímsson, 2012)

For any $\sigma \neq 12\cdots k$, and any $j \geq 0$, the number of σ-avoiders with j inversions is a nondecreasing function of length.

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132-avoiders with exactly 2 inversions

n	0	1	2	3	4	5	6	7	8
number	0	0	0	2	2	2	2	2	2

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n	0	1	2	3	4	$\cdots$
number	0	0	0	1	3	$\cdots$

Growth Rates and Inversions

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For any $\sigma \neq 12\cdots k$, and any $j \geq 0$, the number of σ-avoiders with j inversions is a nondecreasing function of length.

Theorem (Claesson, Jelínek, Steingrímsson, 2012)

If the CJS conjecture holds for $\sigma = 1324$, then

$$L(1324) < \qquad 13.001954.$$

Conjecture (Claesson, Jelínek, Steingrímsson, 2012)

For any $\sigma \neq 12\cdots k$, and any $j \geq 0$, the number of σ-avoiders with j inversions is a nondecreasing function of length.

Theorem (Claesson, Jelínek, Steingrímsson, 2012)

If the CJS conjecture holds for $\sigma = 1324$, then

$$L(1324) < e^{\pi\sqrt{2/3}} \approx 13.001954.$$

The Conway-Guttmann Estimate

Conjecture (Conway and Guttmann, 2014)

There are constants B, μ, μ_1, and g such that

$$|Av_n(1324)| \sim B\mu^n \mu_1^{\sqrt{n}} n^g.$$

The Conway-Guttmann Estimate

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There are constants B, μ, μ_1, and g such that

$$|Av_n(1324)| \sim B\mu^n \mu_1^{\sqrt{n}} n^g.$$

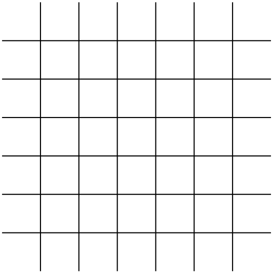
$$\mu = 11.60 \pm 0.01$$

$$\mu_1 = 0.0398 \pm 0.001$$

$$g = -1.1 \pm 0.2$$

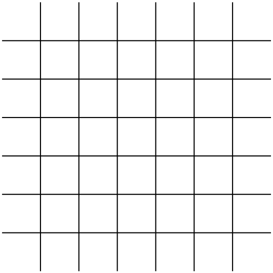
$$B = 9.5 \pm 1.0$$

The Dukes-Parton-West Permutation Patterns Game



- Fix a permutation σ.
- Players take turns placing stones on grid points.

The Dukes-Parton-West Permutation Patterns Game



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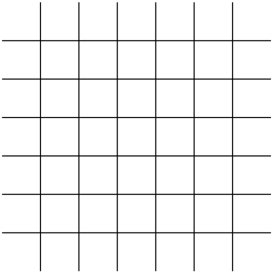

- Fix a permutation σ.
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- No two stones may be in the same row or column.
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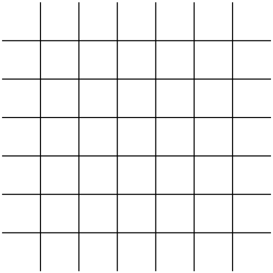
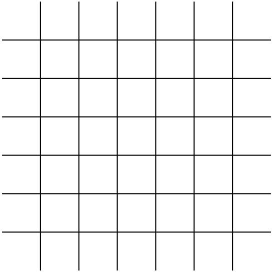
The Dukes-Parton-West Permutation Patterns Game



- Fix a permutation σ.
- Players take turns placing stones on grid points.
- No two stones may be in the same row or column.
- No occurrence of σ allowed.
- Last player to move wins.

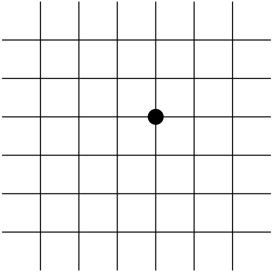
Would You Like to Play a Game?



$\sigma = 21$

Is it better to play first or second?

What If Your Opponent Goes First, But Is Confused?



$\sigma = 21$

Where should you play?

A More Complicated Pattern

If $\sigma = 321$,

should you play

first or second?

Board Size	Winning Player
1×1	
2×2	
3×3	
4×4	
5×5	
6×6	
7×7	
8×8	

A More Complicated Pattern

If $\sigma = 321$,

should you play

first or second?

Board Size	Winning Player
1×1	first
2×2	second
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4×4	
5×5	
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7×7	
8×8	

A More Complicated Pattern

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A More Complicated Pattern

If $\sigma = 321$,

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Board Size	Winning Player
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2×2	second
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5×5	first
6×6	second
7×7	
8×8	

A More Complicated Pattern

If $\sigma = 321$,

should you play

first or second?

Board Size	Winning Player
1×1	first
2×2	second
3×3	first
4×4	second
5×5	first
6×6	second
7×7	first
8×8	first!!!

A More Complicated Pattern

If $\sigma = 321$,

should you play

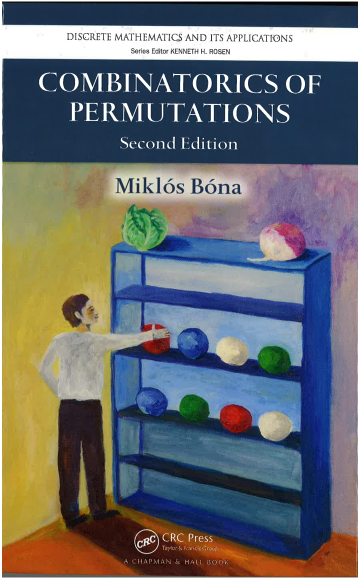
first or second?

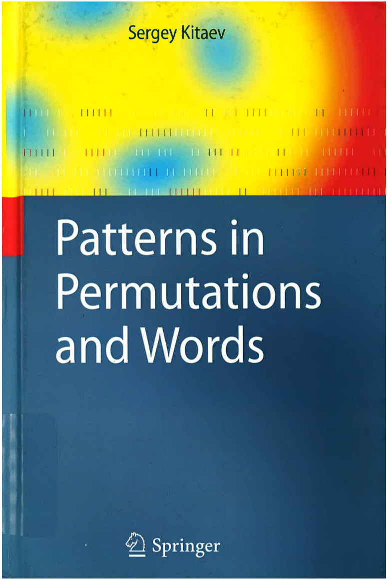
Board Size	Winning Player
1×1	first
2×2	second
3×3	first
4×4	second
5×5	first
6×6	second
7×7	first
8×8	first!!!

Open Problem

Find the general pattern.

Where to Learn More





Two More References

www-circa.mcs.st-and.ac.uk/PermutationPatterns2007/talks/west.pdf

E. Steingrímsson. Some open problems on permutation patterns. In Surveys in Combinatorics, Cambridge University Press, 2013.

The End

Thank You!