Pattern-Avoiding Permutations and Lattice Paths: Old Connections and New Links

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Carleton College

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Permutations and Pattern Avoidance

Definition

π, σ are permutations.
π avoids σ whenever π has no subsequence with same length and relative order as σ.

Example

6152347 avoids 231 but not 213.

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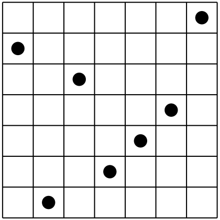

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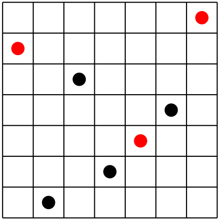

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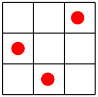

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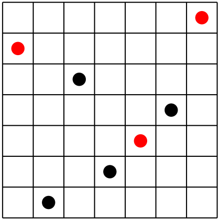
Notation

$Av(\sigma)$:= *set of all permutations which avoid* σ.

$$Av_n(\sigma) = Av(\sigma) \cap S_n$$



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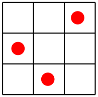

Counting Pattern-Avoiding Permutations

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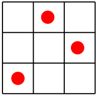
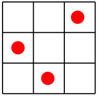
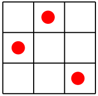
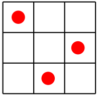
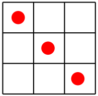
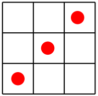
   

$$|Av_n(321)| = |Av_n(123)|$$

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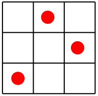
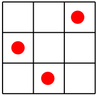
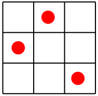
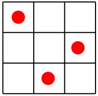

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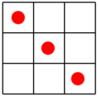
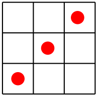

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Rotation of diagrams gives bijections among these sets.

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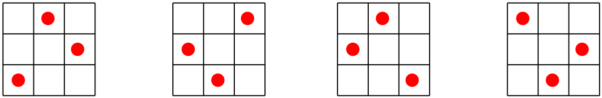

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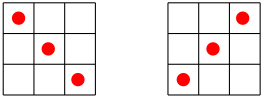

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$$|Av_n(231)| = |Av_n(321)| = C_n = \frac{1}{n+1}\binom{2n}{n}$$

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A *Catalan path* (of length n) is a sequence of n North $(0,1)$ steps and n East $(1,0)$ steps which never passes below the line $y = x$.

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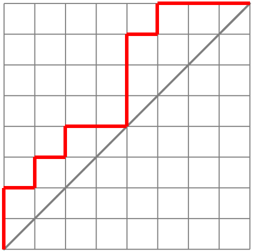

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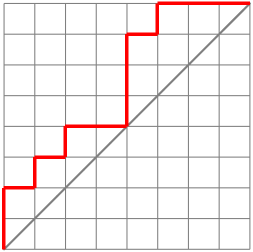

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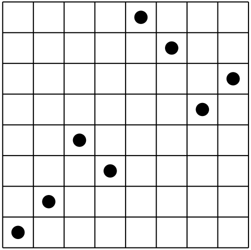
Recursive Structures

Permutations



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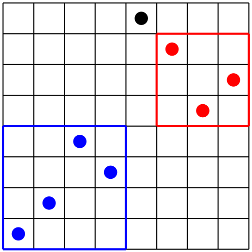

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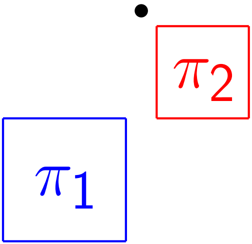

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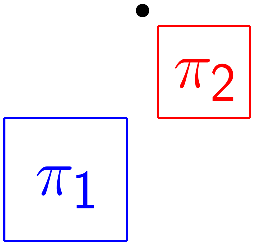

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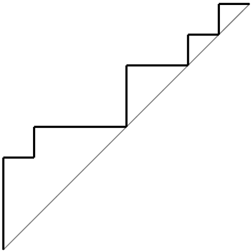

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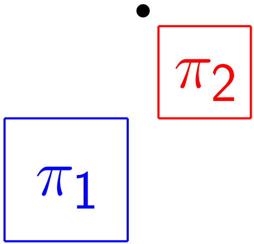

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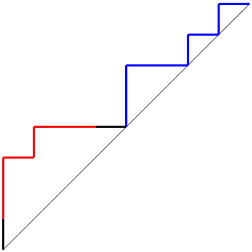

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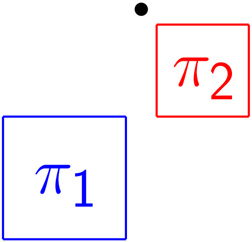

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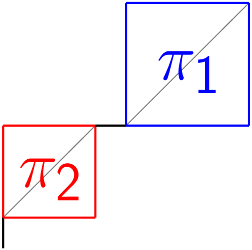

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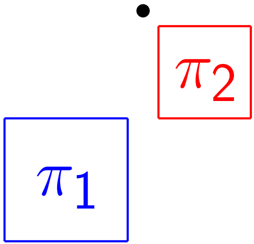

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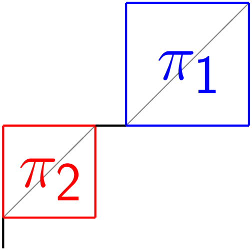

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Idea

$$F(\pi_1 \oplus \pi_2) = N\ F(\pi_2)\ E\ F(\pi_1)$$

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An *inversion* in a permutation is an occurence of the pattern 21.

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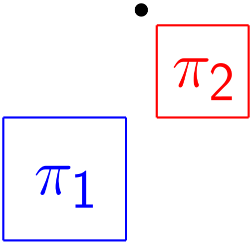

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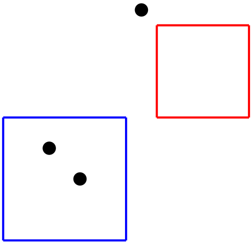

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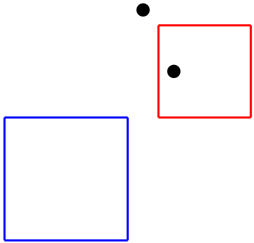

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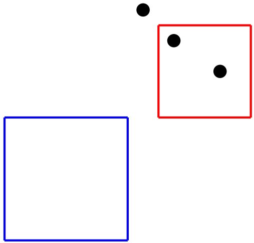

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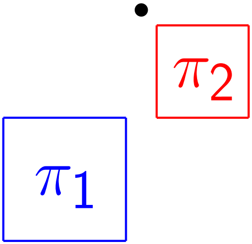

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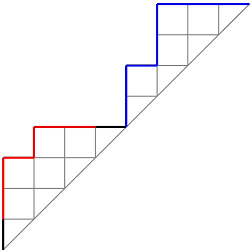

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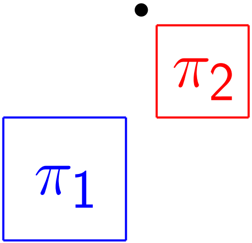

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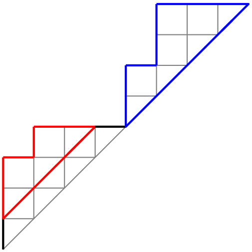

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$$\text{inv}(\pi) = \text{area}(F(\pi))$$

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The *height* $\mathrm{ht}(s)$ of an East step s in a Catalan path π is the number of area squares below it. The *kth area* of π is $\mathrm{area}_k(\pi) = \sum_{s \in \pi} \binom{\mathrm{ht}(s)}{k-1}$.

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Theorem

$$k(\pi) = \mathrm{area}_k(F(\pi))$$

and

$$\sum_{\pi \in Av(231)} x_1^{1(\pi)} x_2^{2(\pi)} x_3^{3(\pi)} \cdots = \cfrac{1}{1 - \cfrac{x_1}{1 - \cfrac{x_1 x_2}{1 - \cfrac{x_1 x_2^2 x_3}{\cdots}}}}.$$

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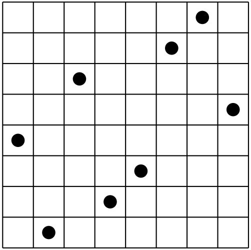

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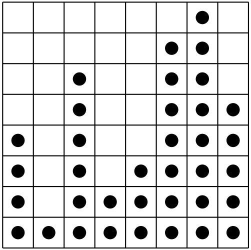

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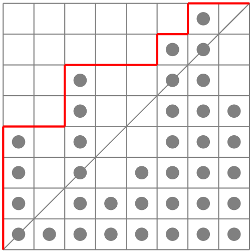

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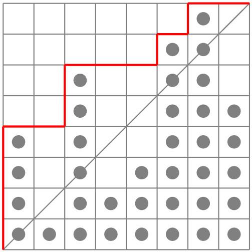

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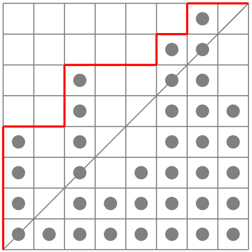

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If the ith East step is below $y = x$ then the first i buildings are all height $i - 1$ or less.

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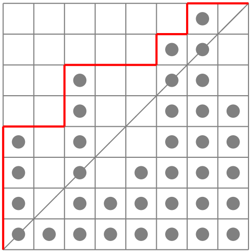

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The restriction to $Av(321)$ is a bijection.

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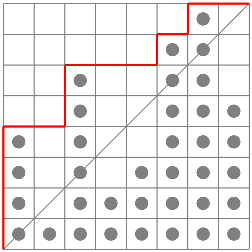

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To avoid 321, we must have increasing heights in the canyons.

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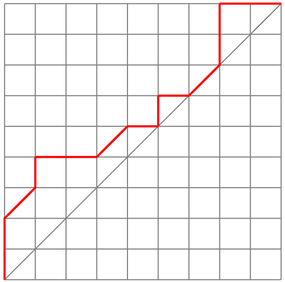

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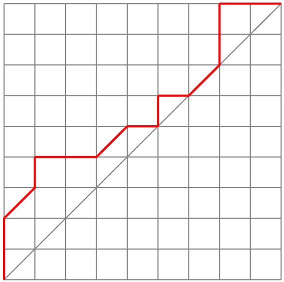

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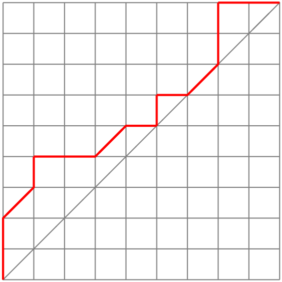
$$r_n = \sum_{d=0}^{n} \binom{2n-d}{d} C_{n-d}$$



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$$|Av_n(3421, 3412)| = r_{n-1}$$

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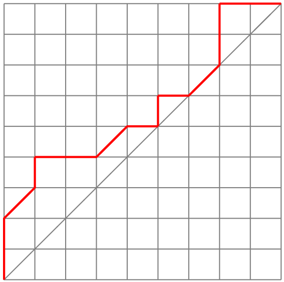

A Schröder Path

$$r_n = \sum_{d=0}^{n} \binom{2n-d}{d} C_{n-d}$$

Theorem

$$|Av_n(3421, 3412)| = r_{n-1}$$

Theorem

$$k(\pi) = \text{area}_k(F(\pi))$$

and

$$\sum_{\pi \in Av(3421,3412)} x_1^{1(\pi)} x_2^{2(\pi)} x_3^{3(\pi)} \cdots = 1 + \cfrac{x_1}{1 - x_1 - \cfrac{x_1 x_2}{1 - x_1 x_2 - \cfrac{x_1 x_2^2 x_3}{\cdots}}}.$$

An Open Schröder Problem

Conjecture

$$|Av_n(2413, 2143, 415263)| = r_{n-1}$$

The End

Thank You!