The Jacobi-Stirling Numbers

Eric S. Egge
(joint work with G. Andrews, L. Littlejohn, and W. Gawronski)

Carleton College

March 18, 2012

The Differential Operator xD

$$y = y(x) \qquad\qquad D = \frac{d}{dx}$$

The Differential Operator xD

$$y = y(x) \qquad\qquad D = \frac{d}{dx}$$

$$xD[y] = 1xy'$$
$$(xD)^2[y] = 1xy' + 1x^2y^{(2)}$$

The Differential Operator xD

$$y = y(x) \qquad\qquad D = \frac{d}{dx}$$

$$\begin{aligned} xD[y] &= 1xy' \\ (xD)^2[y] &= 1xy' + 1x^2y^{(2)} \\ (xD)^3[y] &= 1xy' + 3x^2y^{(2)} + 1x^3y^{(3)} \end{aligned}$$

The Differential Operator xD

$$y = y(x) \qquad D = \frac{d}{dx}$$

$$\begin{aligned}
xD[y] &= 1xy' \\
(xD)^2[y] &= 1xy' + 1x^2y^{(2)} \\
(xD)^3[y] &= 1xy' + 3x^2y^{(2)} + 1x^3y^{(3)} \\
(xD)^4[y] &= 1xy' + 7x^2y^{(2)} + 6x^3y^{(3)} + 1x^4y^{(4)}
\end{aligned}$$

The Differential Operator xD

$$y = y(x) \qquad\qquad D = \frac{d}{dx}$$

$$xD[y] = 1xy'$$
$$(xD)^2[y] = 1xy' + 1x^2y^{(2)}$$
$$(xD)^3[y] = 1xy' + 3x^2y^{(2)} + 1x^3y^{(3)}$$
$$(xD)^4[y] = 1xy' + 7x^2y^{(2)} + 6x^3y^{(3)} + 1x^4y^{(4)}$$

$$(xD)^n[y] = \sum_{j=1}^{n} \left\{ {n \atop n+1-j} \right\} x^j y^{(j)} \qquad \left\{ {n \atop j} \right\} = \left\{ {n-1 \atop j-1} \right\} + j \left\{ {n-1 \atop j} \right\}$$

The Jacobi-Stirling Numbers

$$\ell_{\alpha,\beta}[y] = -(1 - x^2)y'' + (\alpha - \beta + (\alpha + \beta + 2)x)y'$$

$$\ell_{\alpha,\beta}[y] = -(1-x^2)y'' + (\alpha - \beta + (\alpha+\beta+2)x)y'$$

Theorem (Everitt, Kwon, Littlejohn, Wellman, Yoon)

$$\ell^n_{\alpha,\beta}[y] = \frac{1}{w_{\alpha,\beta}(x)} \sum_{j=1}^{n} \left\{ {n \atop j} \right\}_{\alpha,\beta} (-1)^j \left((1-x)^{\alpha+j}(1+x)^{\beta+j} y^{(j)} \right)^{(j)}$$

$$\ell_{\alpha,\beta}[y] = -(1-x^2)y'' + (\alpha - \beta + (\alpha+\beta+2)x)y'$$

Theorem (Everitt, Kwon, Littlejohn, Wellman, Yoon)

$$\ell^n_{\alpha,\beta}[y] = \frac{1}{w_{\alpha,\beta}(x)} \sum_{j=1}^{n} \left\{ {n \atop j} \right\}_{\alpha,\beta} (-1)^j \left((1-x)^{\alpha+j}(1+x)^{\beta+j} y^{(j)} \right)^{(j)}$$

$$\left\{ {n \atop j} \right\}_{\alpha,\beta} = \left\{ {n-1 \atop j-1} \right\}_{\alpha,\beta} + j(j+\alpha+\beta+1) \left\{ {n-1 \atop j} \right\}_{\alpha,\beta}$$

What Does $\left\{ {n \atop j} \right\}_{\alpha,\beta}$ Count?

$$2\gamma - 1 = \alpha + \beta + 1$$

$$[n]_2 := \{1_1, 1_2, 2_1, 2_2, \ldots, n_1, n_2\}$$

What Does $\left\{ {n \atop j} \right\}_{\alpha,\beta}$ Count?

$$2\gamma - 1 = \alpha + \beta + 1 \qquad\qquad [n]_2 := \{1_1, 1_2, 2_1, 2_2, \ldots, n_1, n_2\}$$

Theorem (AEGL)

For any positive integer γ, the Jacobi-Stirling number $\left\{ {n \atop j} \right\}_{\gamma}$ counts set partitions of $[n]_2$ into $j + \gamma$ blocks such that

1. *There are γ distinguishable zero blocks, any of which may be empty.*
2. *There are j indistinguishable nonzero blocks, all nonempty.*
3. *The union of the zero blocks does not contain both copies of any number.*
4. *Each nonzero block*
   - *contains both copies of its smallest element*
   - *does not contain both copies of any other number.*

These are Jacobi-Stirling set partitions.

A Generating Function Interpretation

$$z = \alpha + \beta$$

A Generating Function Interpretation

$$z = \alpha + \beta$$

$$S(n, j) := \text{Jacobi-Stirling set partitions of } [n]_2 \text{ into 1 zero block and } j \text{ nonzero blocks.}$$

A Generating Function Interpretation

$$z = \alpha + \beta$$

$$S(n,j) := \text{Jacobi-Stirling set partitions of } [n]_2$$
$$\text{into 1 zero block and } j \text{ nonzero blocks.}$$

Theorem (Gelineau, Zeng)

The Jacobi-Stirling number $\left\{ {n \atop j} \right\}_z$ is the generating function in z for $S(n,j)$ with respect to the number of numbers with subscript 1 in the zero block.

A Generating Function Interpretation

$$z = \alpha + \beta$$

$$S(n,j) := \text{Jacobi-Stirling set partitions of } [n]_2 \text{ into 1 zero block and } j \text{ nonzero blocks.}$$

Theorem (Gelineau, Zeng)

The Jacobi-Stirling number $\left\{ {n \atop j} \right\}_z$ *is the generating function in* z *for* $S(n,j)$ *with respect to the number of numbers with subscript* 1 *in the zero block.*

Corollary

The leading coefficient in $\left\{ {n \atop j} \right\}_z$ *is the Stirling number* $\left\{ {n \atop j} \right\}$.

Jacobi-Stirling Numbers of the First Kind

$$x^n = \sum_{j=0}^{n} \left\{ {n \atop j} \right\}_{\alpha,\beta} \prod_{k=0}^{j-1} (x - k(k+\alpha+\beta+1))$$

Jacobi-Stirling Numbers of the First Kind

$$x^n = \sum_{j=0}^{n} \left\{ {n \atop j} \right\}_{\alpha,\beta} \prod_{k=0}^{j-1} (x - k(k+\alpha+\beta+1))$$

$$\prod_{k=0}^{n-1} (x - k(k+\alpha+\beta+1)) = \sum_{j=0}^{n} (-1)^{n+j} \left[{n \atop j} \right]_{\alpha,\beta} x^j$$

Jacobi-Stirling Numbers of the First Kind

$$x^n = \sum_{j=0}^{n} \left\{ {n \atop j} \right\}_{\alpha,\beta} \prod_{k=0}^{j-1} (x - k(k+\alpha+\beta+1))$$

$$\prod_{k=0}^{n-1} (x - k(k+\alpha+\beta+1)) = \sum_{j=0}^{n} (-1)^{n+j} \left[{n \atop j} \right]_{\alpha,\beta} x^j$$

$$\left[{n \atop j} \right]_{\alpha,\beta} = \left[{n-1 \atop j-1} \right]_{\alpha,\beta} + (n-1)(n+\alpha+\beta) \left[{n-1 \atop j} \right]_{\alpha,\beta}$$

Jacobi-Stirling Numbers of the First Kind

$$x^n = \sum_{j=0}^{n} \left\{ {n \atop j} \right\}_{\alpha,\beta} \prod_{k=0}^{j-1} (x - k(k+\alpha+\beta+1))$$

$$\prod_{k=0}^{n-1} (x - k(k+\alpha+\beta+1)) = \sum_{j=0}^{n} (-1)^{n+j} \left[{n \atop j} \right]_{\alpha,\beta} x^j$$

$$\left[{n \atop j} \right]_{\alpha,\beta} = \left[{n-1 \atop j-1} \right]_{\alpha,\beta} + (n-1)(n+\alpha+\beta) \left[{n-1 \atop j} \right]_{\alpha,\beta}$$

Theorem (AEGL)

$$\left\{ {-j \atop -n} \right\}_{\gamma} = (-1)^{n+j} \left[{n \atop j} \right]_{1-\gamma}$$

Balanced Jacobi-Stirling Permutations

Theorem (AEGL)

For any positive integer γ, the Jacobi-Stirling number of the first kind $\left[{n \atop j}\right]_\gamma$ is the number of ordered pairs (π_1, π_2) of permutations with $\pi_1 \in S_{n+\gamma}$ and $\pi_2 \in S_{n+\gamma-1}$ such that

1. *π_1 has $\gamma + j$ cycles and π_2 has $\gamma + j - 1$ cycles.*
2. *The cycle maxima of π_1 which are less than $n + \gamma$ are exactly the cycle maxima of π_2.*
3. *For each non cycle maximum k, at least one of $\pi_1(k)$ and $\pi_2(k)$ is less than or equal to n.*

Such ordered pairs are balanced Jacobi-Stirling permutations.

Unbalanced Jacobi-Stirling Permutations

Theorem (AEGL)

If 2γ is a positive integer, then the Jacobi-Stirling number of the first kind $\left[{n \atop j}\right]_\gamma$ is the number of ordered pairs (π_1, π_2) of permutations with $\pi_1 \in S_{n+\gamma}$ and $\pi_2 \in S_n$ such that

1. *π_1 has $j + 2\gamma - 1$ cycles and π_2 has j cycles.*
2. *The cycle maxima of π_1 which are less than $n + 1$ are exactly the cycle maxima of π_2.*
3. *For each non cycle maximum k, at least one of $\pi_1(k)$ and $\pi_2(k)$ is less than or equal to n.*

Such ordered pairs are unbalanced Jacobi-Stirling permutations.

More Generating Functions

$\Sigma(n, j) :=$ all (σ, τ) such that

- σ is a permutation of $\{0, 1, \ldots, n\}$, τ is a permutation of $\{1, 2, \ldots, n\}$, and both have j cycles.
- 1 and 0 are in the same cycle in σ.
- Among their nonzero entries, σ and τ have the same cycle minima.

More Generating Functions

$\Sigma(n, j) :=$ all (σ, τ) such that

- σ is a permutation of $\{0, 1, \ldots, n\}$, τ is a permutation of $\{1, 2, \ldots, n\}$, and both have j cycles.
- 1 and 0 are in the same cycle in σ.
- Among their nonzero entries, σ and τ have the same cycle minima.

Theorem (Gelineau, Zeng)

$\left[{n \atop j} \right]_z$ *is the generating function in* z *for* $\Sigma(n, j)$ *with respect to the number of nonzero left-to-right minima in the cycle containing 0 in* σ*, written as a word beginning with* $\sigma(0)$*.*

Where are the Symmetric Functions?

$$h_{n-j}(x_1, \ldots, x_j) = h_{n-j}(x_1, \ldots, x_{j-1}) + x_j h_{n-j-1}(x_1, \ldots, x_j)$$

Where are the Symmetric Functions?

$$h_{n-j}(x_1, \ldots, x_j) = h_{n-j}(x_1, \ldots, x_{j-1}) + x_j h_{n-j-1}(x_1, \ldots, x_j)$$

$$\left\{ {n \atop j} \right\}_z = h_{n-j}(1(1+z), 2(2+z), \ldots, j(j+z))$$

Where are the Symmetric Functions?

$$h_{n-j}(x_1,\ldots,x_j) = h_{n-j}(x_1,\ldots,x_{j-1}) + x_j h_{n-j-1}(x_1,\ldots,x_j)$$

$$\left\{ {n \atop j} \right\}_z = h_{n-j}(1(1+z), 2(2+z), \ldots, j(j+z))$$

$$e_{n-j}(x_1,\ldots,x_{n-1}) = e_{n-j}(x_1,\ldots,x_{n-2}) + x_{n-1} e_{n-j-1}(x_1,\ldots,x_{n-2})$$

Where are the Symmetric Functions?

$$h_{n-j}(x_1,\dots,x_j) = h_{n-j}(x_1,\dots,x_{j-1}) + x_j h_{n-j-1}(x_1,\dots,x_j)$$

$$\left\{ {n \atop j} \right\}_z = h_{n-j}(1(1+z), 2(2+z), \dots, j(j+z))$$

$$e_{n-j}(x_1,\dots,x_{n-1}) = e_{n-j}(x_1,\dots,x_{n-2}) + x_{n-1} e_{n-j-1}(x_1,\dots,x_{n-2})$$

$$\left[{n \atop j} \right]_z = e_{n-j}(1(1+z), 2(2+z), \dots, (n-1)(n-1+z))$$

An Open q-uestion

Is there a q-analogue of the Jacobi-Stirling numbers associated with the q-Jacobi polynomials?

The End

Thank You!