Catalan Combinatorics of Borel Ideals and Generalizations

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Borel Ideals

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$B(n) :=$ set of upper triangular matrices in GL_n

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GL_n has a natural action on $\mathbb{C}[x_1, \ldots, x_n]$, so $B(n)$ does, too.

Definition

A *Borel ideal* is an ideal in $\mathbb{C}[x_1, \ldots, x_n]$ which is closed under the action of $B(n)$.

Catalan Combinatorics of Borel Ideals

Theorem (Francisco, Mermin, and Schweig)

The Borel ideal generated by $x_1 x_2 \cdots x_n$ has a minimal generating set (as an ordinary ideal) of C_n monomials.

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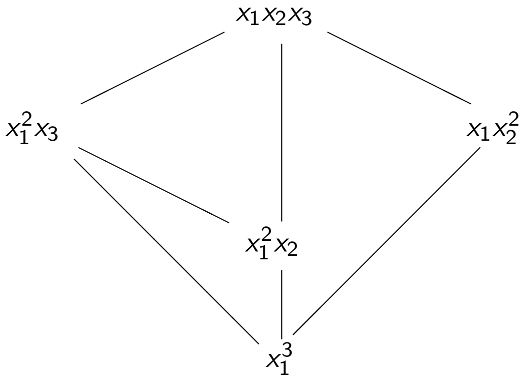
Idea:

$$x_i \mapsto x_j \qquad j < i$$

transforms every generating monomial to another generating monomial.

Catalan Combinatorics of Borel Ideals



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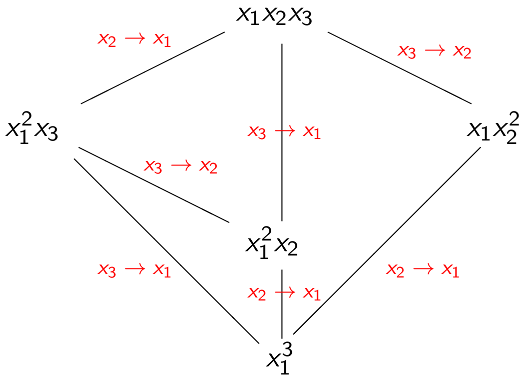

Bijection with Catalan Paths

Observation: The minimal generators are the monomials of degree n whose total degree in $x_1, \ldots, x_j$ is at least j for all j.

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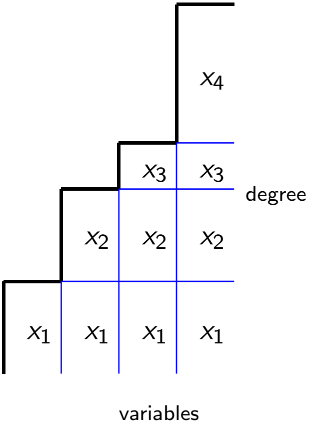
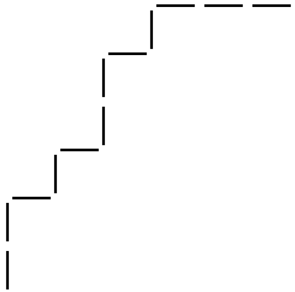


Bijection with Catalan Path Example

$x_1^2 x_2 x_3^2 x_4 \quad \mapsto$



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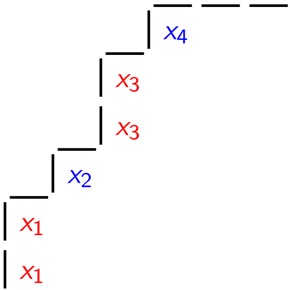

Betti Numbers of Borel Ideals

$$C_{n,k} := \text{number of minimal generators of } \langle x_1 x_2 \cdots x_n \rangle_B$$
$$\text{with largest variable } x_k$$

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$$C_{n,k} = \frac{n-k+1}{n} \binom{n+k-2}{k-1}$$

Betti Numbers of Borel Ideals

Theorem (Francisco, Mermin, and Schweig)

The jth Betti number $b_{n,j}$ of $\langle x_1 x_2 \cdots x_n \rangle_B$ is the number of ordered pairs (m, α) such that

- *m is a minimal generator and*
- *α is a square free monomial of degree j whose largest variable is less than the largest variable of m.*

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Corollary

$$b_{n,j} = \sum_{k=1}^{n} C_{n,k} \binom{k-1}{j}$$

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Corollary

$$b_{n,j} = \frac{1}{n} \binom{2n}{n-j-1} \binom{n+j-1}{j}$$

Combinatorics of $b_{n,j}$: Leaf-Marked Trees

Theorem (Francisco, Mermin, and Schweig)

$b_{n,j}$ is the number of binary trees with

- *j marked leaves and*
- *n unmarked vertices,*

in which the rightmost leaf is not marked.

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Combinatorics of $b_{n,j}$: Branch-Marked Trees

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Combinatorics of $b_{n,j}$: North-Marked Catalan Paths

Theorem (Egge, Rubin)

$b_{n,j}$ is the number of Catalan paths with

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Combinatorics of $b_{n,j}$: 132-Avoiding Permutations

Theorem (Egge)

$b_{n,j}$ is the number of 132-avoiding permutations with

- *n unbarred entries,*
- *j barred entries,*
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- *every barred entry is a local minimum.*

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$\bar{2}314 \qquad \bar{2}341 \qquad 3\bar{2}41 \qquad \bar{3}412 \qquad \bar{3}421 \qquad 4\bar{2}31$

Combinatorics of $b_{n,j}$: 321-Avoiding Permutations

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132   213   231   231   312   312

Combinatorics of $b_{n,j}$: Triangulations

Theorem (Egge)

$b_{n,j}$ is the number of

- *triangulations of an $n + j + 2$-gon,*
- *with j shaded triangles with two edges on the boundary,*
- *in which the triangle adjacent to the bottom edge is not shaded*
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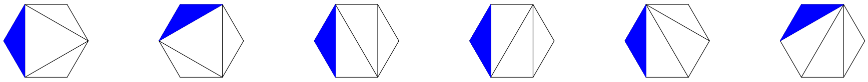

Combinatorics of $b_{n,j}$: Noncrossing Partitions

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$1/\overline{2}34 \qquad 12/\overline{3}4 \qquad 14/\overline{2}3 \qquad 1/\overline{2}3/4 \qquad 1/\overline{2}4/3 \qquad 1/2/\overline{3}4$

Conjectured Combinatorics of $b_{n,j}$: Dumont Permutations

Definition

A *Dumont permutation (of the first kind)* is a permutation in which every even entry is followed by a descent, each odd entry is followed by an ascent, and the last entry is odd.

Theorem (Burstein)

The number of Dumont permutations of length $2n$ *which avoid 2413 and 3142 is*

$$\sum_{j=0}^{n-1} b_{n,j}.$$

Conjectured Combinatorics of $b_{n,j}$: Rotationally Symmetric Permutations

Conjecture (Egge)

The number of rotationally symmetric permutations of length 4n which avoid 2413 is

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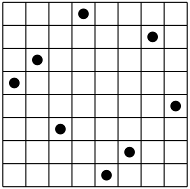

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A k-ary Generalization

Theorem (Egge)

The Borel ideal generated by

$$x_1 \; x_{1+k} \; x_{1+2k} \cdots x_{1+(n-1)k}$$

has a minimal generating set of

$$\frac{1}{(k-1)n+1}\binom{kn}{n}$$

monomials.

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The End

Thank You!