A Chromatic Symmetric Function for Signed Graphs

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March 5, 2016

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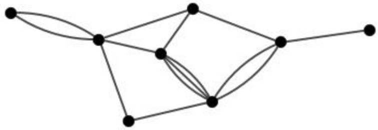

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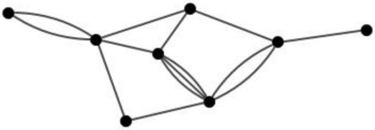

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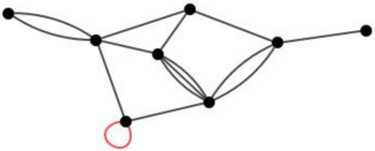

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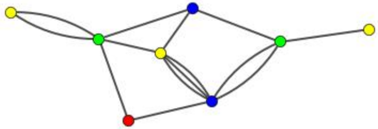

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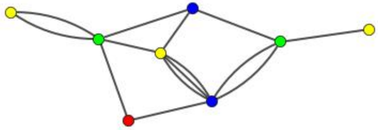

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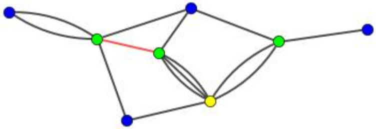

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Definition (Stanley)

The chromatic symmetric function of G is

$$X_G = \sum_{C \text{ proper coloring of } G} x(C).$$

Signed Graphs

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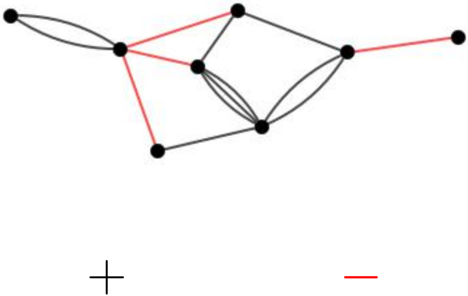

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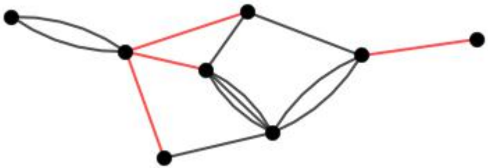

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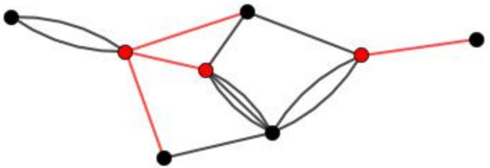

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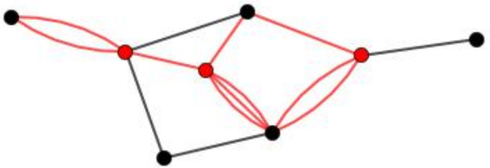

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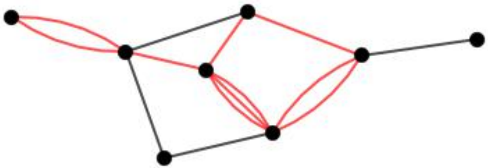

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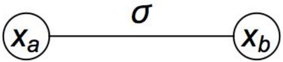

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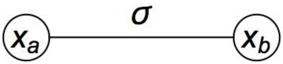

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Fact

If G and H are related by switching then there is a natural bijection between their sets of proper colorings.

The Chromatic Symmetric Function of a Signed Graph

Definition

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Observation

Y_G is invariant under the natural action of the hyperoctahedral group, which is the set of permutations π of $\pm 1, \pm 2, \ldots$ such that

$$\pi(-j) = -\pi(j)$$

for all j.

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$$Y_G \in BSym$$

Marked Ferrers Diagrams

Goal: a basis for *BSym*.

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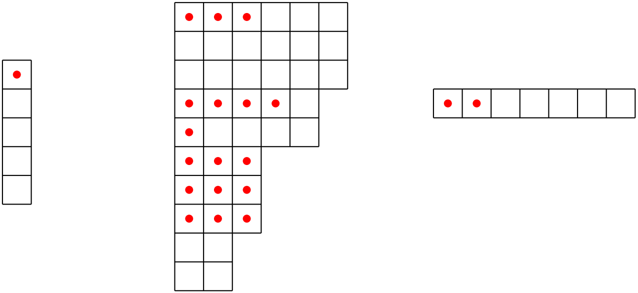
$$|\lambda| := \text{total number of boxes and dots in } \lambda$$

Marked Ferrers Diagrams and Their Monomials



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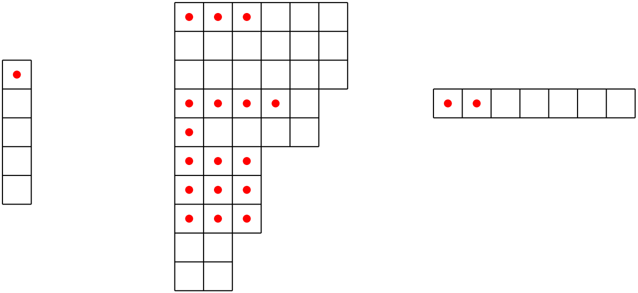
For each marked Ferrers diagram there is a monomial.



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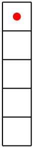
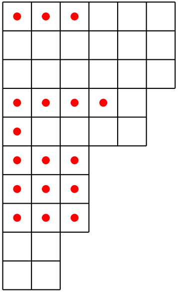


$x_1 x_{-1} x_2 x_3 x_4 x_5$



$x_1^6 x_{-1}^3 x_2^6 x_3^6 x_4^5 x_{-4}^4 \cdots$

$x_1^7 x_{-1}^2$

A *BSym* Basis

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For any marked Ferrers diagram λ, m_λ is the sum of the distinct images of λ's monomial.

Theorem

$$\{m_\lambda \mid |\lambda| = n\} \text{ is a basis for } BSym_n.$$

dim $BSym_n$

n	0	1	2	3	4	5	6	7	8
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Theorem

$$\sum_{n=0}^{\infty} \dim(BSym_n)x^n = \prod_{j=1}^{\infty} \left(\frac{1}{1-x^j}\right)^{\lfloor j/2 \rfloor + 1}$$

The Power Sum Basis

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Theorem

If we linearly order the set of row shapes then

$$\{p_{\lambda_1,\ldots,\lambda_k} \mid \sum_j |\lambda_j| = n \text{ and } \lambda_1 \geq \cdots \geq \lambda_k\}$$

is a basis for $BSym_n$.

The Elementary Basis?

$e_\lambda := m_\lambda$ for any λ with just one column

$$e_{\lambda_1,\ldots,\lambda_k} := e_{\lambda_1} \cdots e_{\lambda_k}$$

for any list $\lambda_1, \ldots, \lambda_k$ of column shapes

Conjecture

If we linearly order the set of column shapes then

$$\{e_{\lambda_1,\ldots,\lambda_k} \mid \sum_j |\lambda_j| = n \text{ and } \lambda_1 \geq \cdots \geq \lambda_k\}$$

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Basic Results: The Chromatic Polynomial

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Theorem

If G is a signed graph then

$$Y_G(\underbrace{1, 1, \ldots, 1}_{n}, 0, 0, \ldots) = \chi_G(n)$$

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If a signed graph G is a disjoint union of signed graphs G_1 and G_2 then

$$Y_G = Y_{G_1} \cdot Y_{G_2}.$$

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Theorem

If all of the edges in a signed graph G are positive then

$$Y_G = X_G(x_1, x_{-1}, x_2, x_{-2}, \ldots).$$

Switching Does Not Preserve Y_G

$m_{\boxdot} + 2m_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$

$m_{\square\square} + 2m_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$

The Power Basis Expansion

Definition

For any connected, signed graph G, the type $\lambda(G)$ of G is the row shape consisting of k boxes and m dots, where G can be colored with k x_1s and m x_{-1}s so that every edge is improper. If G is not connected then its type is the sequence of types of its connected components.

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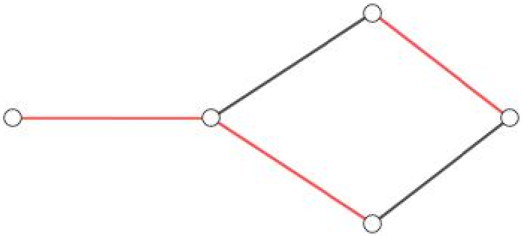

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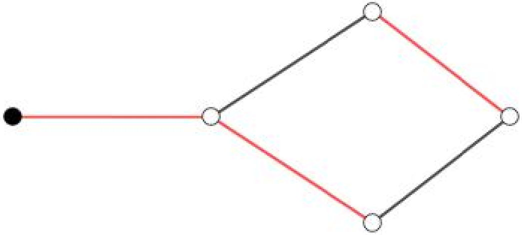

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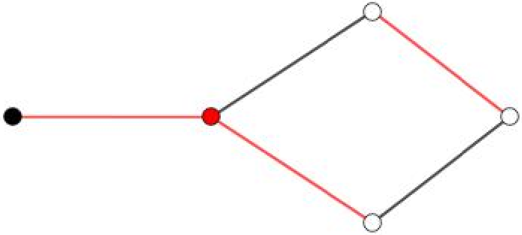

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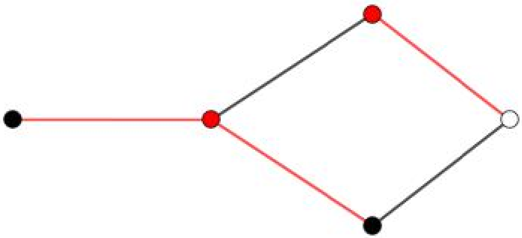

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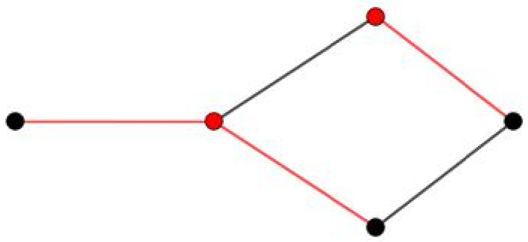

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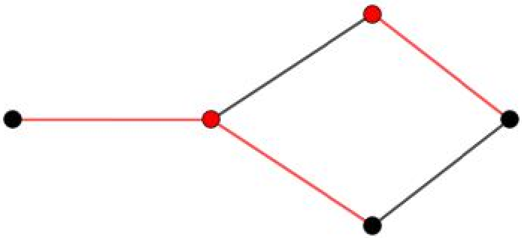
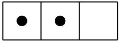


$\lambda(G) =$ 

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A connected signed graph G is 2-faced whenever there are two colorings of its vertices with x_1 and x_{-1} which are improper along every edge, and which have at least as many x_1s as x_{-1}s.

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Example

Every path with an even number of vertices is 2-faced.

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Every path with an even number of vertices is 2-faced.

Example

Every cycle with an even number of vertices whose product of signs is positive is 2-faced.

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Theorem

For any signed graph G with edge set E,

$$Y_G = \sum_{S \subseteq E} (-1)^{|S|} 2^{tf(S)} p_{\lambda(S)},$$

where $tf(S)$ is the number of 2-faces of S and $p_{\lambda(S)} = 0$ if S has no type.

The Last Slide

Thank you!