A New Notion of Noncontiguous Containment for Ordered, Rooted Trees

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Carleton College

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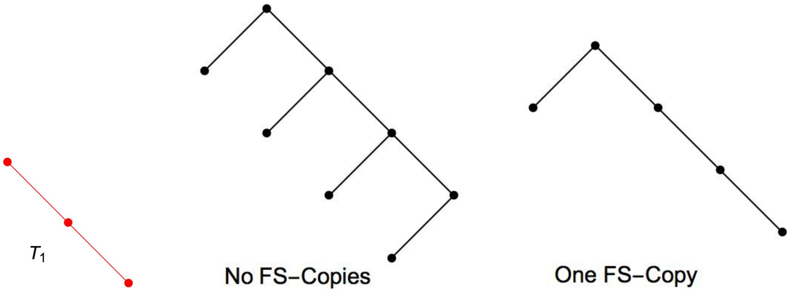
Outline

1. Containment Notions for Binary Trees
2. Enumerations Involving C-Containment and C-Avoidance
3. Connections with Pattern Avoidance in Permutations

FS-Containment

Flajolet and Sedgewick (2009):

An FS-copy of T_1 in T_2 is a node in T_1 whose dangling subtree (including all of its children) is isomorphic to T_1.



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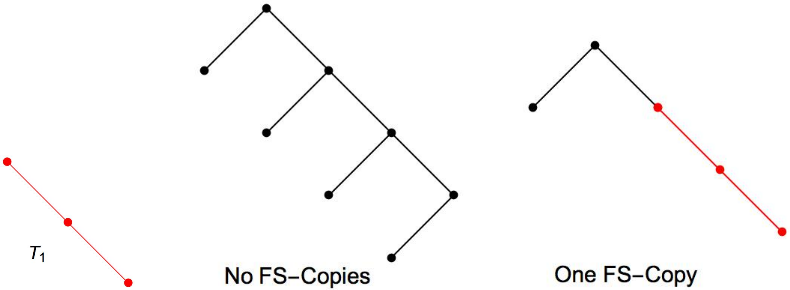

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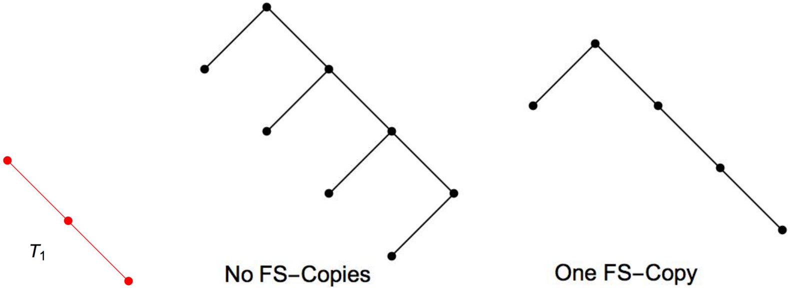


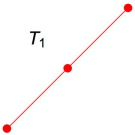
Entire subtree of chosen root must match.

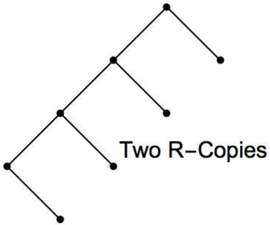
Removing vertices can create a copy.

R-Containment

Rowland (2010):

An R-copy of T_1 in T_2 is a connected copy of T_1 in T_2.





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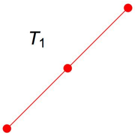

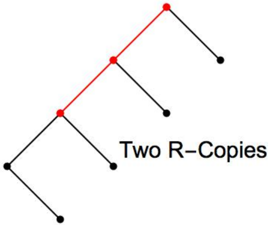

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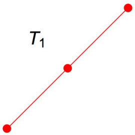

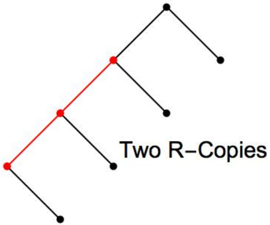

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P-Containment

Dairyko, Pudwell, Tyner, Wynn (2012) and Pudwell, Scholten, Schrock, and Serrato (2014):

For full trees (no node has exactly 1 child), a P-copy of T_1 in T_2 is a set E of edges in T_2 such that

- for each node v in T_2,
  both edges from v are in E or neither is,
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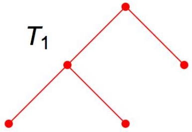
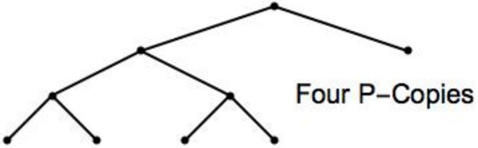
  and
- the tree left after contracting all edges not in E is isomorphic to T_1.

A P-Containment Example





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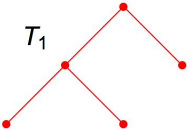

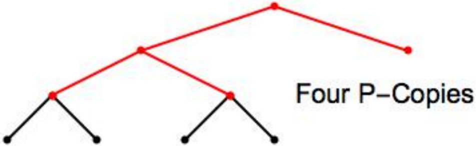

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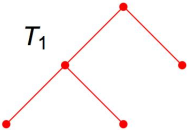

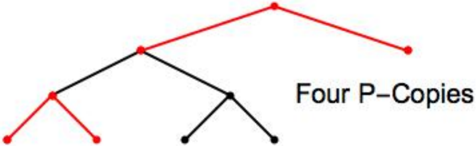

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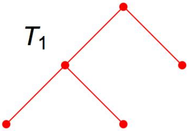

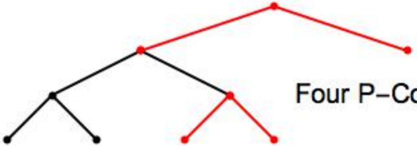

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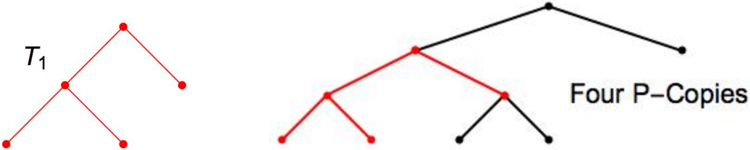

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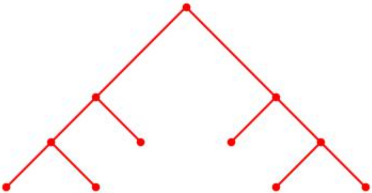

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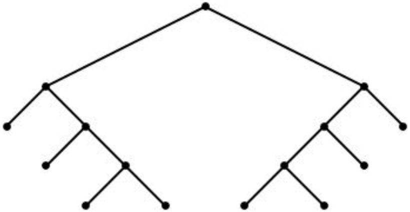

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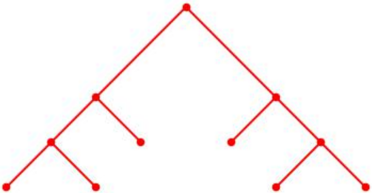

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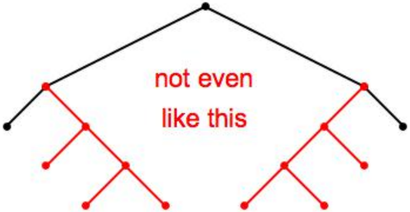

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3. For all $s, t, u \in T$, if $s < t$ and $t <_L u$ then $s <_L u$. Similarly, if $s < t$ and $t <_R u$ then $s <_R u$.
4. For all $s, t \in T$, at most one of the following holds: $s <_L t$, $s <_R t$, $t <_L s$, and $t <_R s$.
5. For all $s \in T$, if the set of all $t \in T$ with $t <_L s$ is nonempty, then there is a unique $u \in T$ such that $u <_L s$ and if $t \neq u$ has $t <_L s$ then $t < u$.
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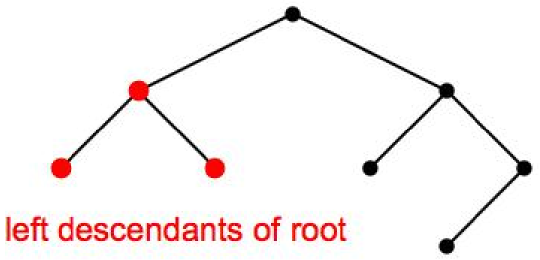

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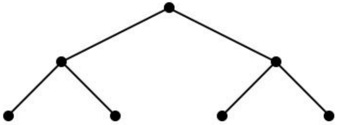

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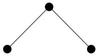

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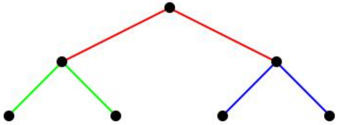

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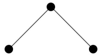

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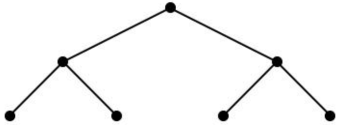

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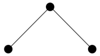

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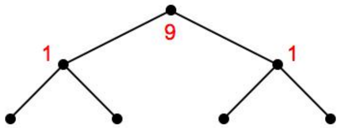

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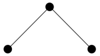

C-Avoidance and P-Avoidance

avoids := contains no copy of

Theorem (Egge)

For any full binary trees T_1 and T_2, T_2 C-avoids T_1 if and only if T_2 P-avoids T_1.

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In a C-copy, move chosen non-root vertices with unchosen partners up until siblings are chosen in pairs and keep all edges up from nonroot chosen vertices.

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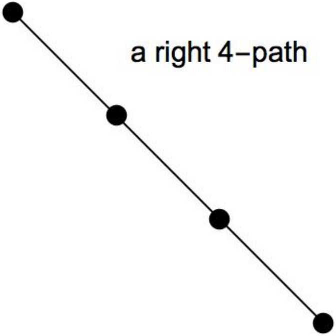
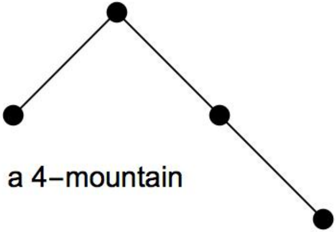
Paths and Mountains





C-Avoiding Paths

Theorem (Egge)

If $F_k(x)$ is the ogf for binary trees with n vertices C-avoiding a right k-path then

$$F_k(x) = \frac{U_{k-1}\left(\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}U_k\left(\frac{1}{2\sqrt{x}}\right)},$$

where

$$U_k(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} (2x)^{n-2k}$$

is the kth Chebyshev polynomial of the second kind.

C-Avoiding Mountains

Theorem (Egge)

For any k-mountain M, the number of binary trees with n vertices C-avoiding M is equal to the number of binary trees with n vertices C-avoiding a right k-path.

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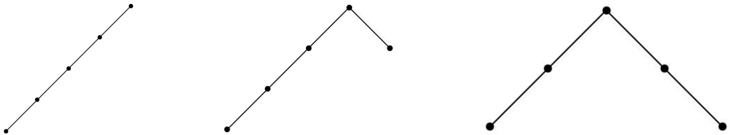
Sketch of proof: compute the generating function recursively and use identities for the Chebyshev polynomials.

Every Tree is (Equivalent to) a Mountain

The same number of trees of each size avoid each of



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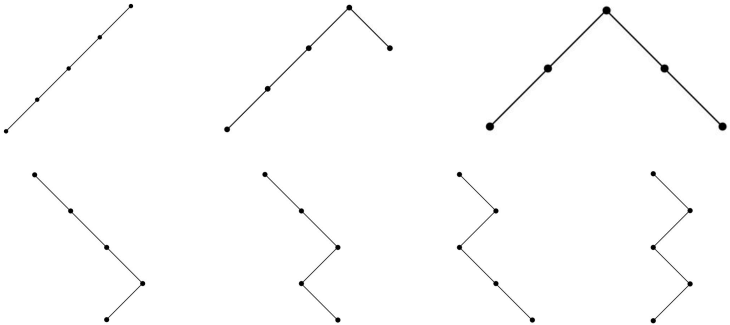

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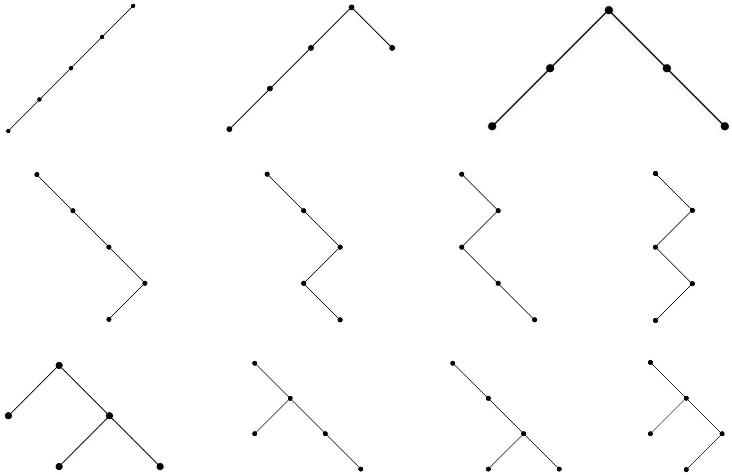

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Theorem (Dairyko, Pudwell, Tyner, and Wynn)

The number of full binary trees with $2n+1$ vertices which P-avoid T depends only on the number of vertices in T.

Counting C-Copies of Complete Trees

Problem: Given a binary tree, how many C-copies of the complete binary tree with k levels does it contain?

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Theorem (Egge)

There are exactly

$$2 \cdot 4^k - (2k+1)2^k - 1$$

copies of the complete binary tree with two levels (and three vertices) in the complete binary tree with k levels.

C-Copies and Continued Fractions

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Observation: C-avoidance generalizes to ternary trees, ordered trees, etc.

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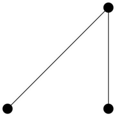
Theorem (Egge)

The number of ternary trees with n vertices C-avoiding both



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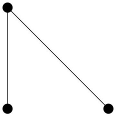

is the number of Motzkin paths with $n-1$ steps in which each level step is one of three colors.

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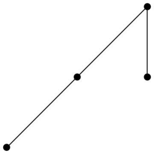

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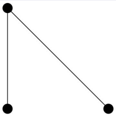

is

$$\frac{1}{n}\sum_{k=0}^{n}\binom{2k+1}{k}\binom{2n}{n-k}\frac{k}{2k+1}.$$

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Binary Trees and 231-Avoiding Permutations

Fact: There is a bijection between binary trees with n vertices and 231-avoiding permutations on $1, 2, \ldots, n$.

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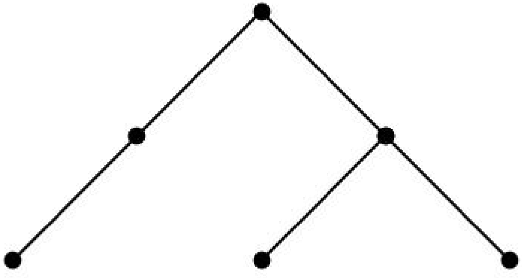
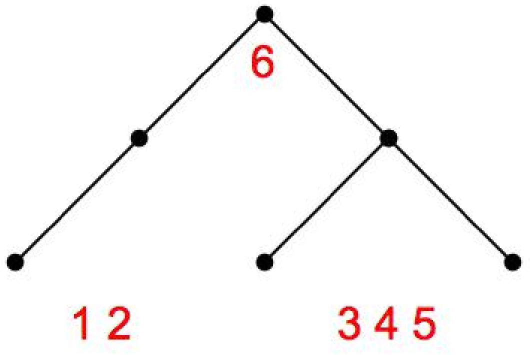
- label the (current) root with the largest available number
- send smaller numbers to the left, larger numbers to the right, and label recursively
- once every vertex is labelled
    - list the labels in the left subtree recursively
    - list the root's label
    - list the labels in the right subtree recursively.

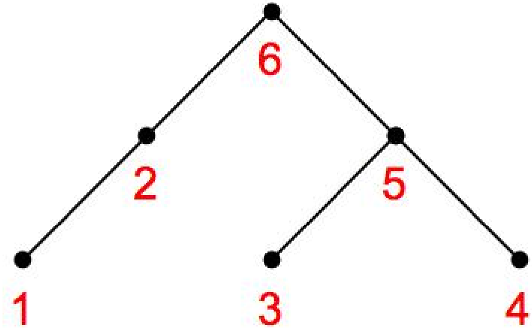
The Bijection in Action



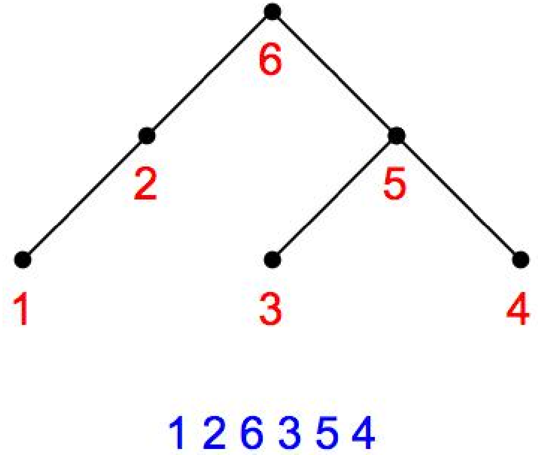
The Bijection in Action



The Bijection in Action



The Bijection in Action



The Permutations for Full Trees

Theorem (Egge)

Under our bijection the full binary trees are in bijection with 231-avoiding permutations which

- *start with an ascent*
- *end with a descent and*
- *have alternating ascents and descents.*

C-Avoidance in Trees and Pattern Avoidance in Permutations

Theorem (Egge)

T_1 and T_2 are binary trees with associated permutations π_1 and π_2. If π_1 avoids π_2 then T_1 C-avoids T_2.

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The converse is false.

C-Avoidance in Trees and Pattern Avoidance in Permutations

Theorem (Egge)

T_1 and T_2 are binary trees with associated permutations π_1 and π_2. If π_1 avoids π_2 then T_1 C-avoids T_2.

The converse is false.

Problem: find conditions under which the converse holds.

The Last Slide

Thank you!