Linear Recurrences and the Pfaffian Transform

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Functions on Sequences

The Binomial Transform

$$B(\{a_j\})_n = \sum_{j=0}^{n} \binom{n}{j} a_j$$

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Theorem (Layman, 2001)

$$H(B(\{a_j\})) = H(\{a_j\})$$

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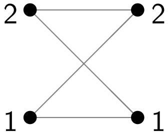

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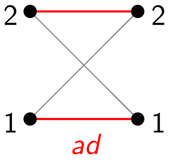

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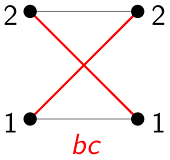

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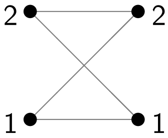

$$\det A = \sum_{\pi \ pm \ K_{n,n}} (-1)^{\mathrm{cross}(\pi)} \prod_{i,j \in \pi} A_{ij}$$

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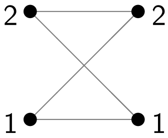

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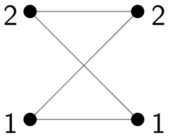

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Fact: $\det(A) = (\operatorname{Pf}(A))^2$

The Pfaffian Transform

$$\mathrm{Pf}(\{a_j\})_n = \mathrm{Pf}\begin{pmatrix} 0 & a_1 & a_2 & a_3 & \cdots & a_{2n-1} \\ -a_1 & 0 & a_1 & a_2 & \cdots & a_{2n-2} \\ -a_2 & -a_1 & 0 & a_1 & \cdots & a_{2n-3} \\ -a_3 & -a_2 & -a_1 & 0 & \cdots & a_{2n-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{2n-1} & -a_{2n-2} & -a_{2n-3} & -a_{2n-4} & \cdots & 0 \end{pmatrix}$$

Pfaffian Transform Examples

$\{a_j\}$	$\mathrm{Pf}(\{a_j\})$
$1, 2, 4, \ldots, 2^{n-1}, \ldots$	$1, 1, 1, \ldots$
$1, 3, 9, \ldots, 3^{n-1}, \ldots$	$1, 1, 1, \ldots$
$1, 1, 2, 3, \ldots, F_n, \ldots$	$1, 2, 4, \ldots, 2^{n-1}, \ldots$
$1, 1, 3, 5, 11, 21, \ldots, J_n, \ldots$	$1, 3, 9, \ldots, 3^{n-1}, \ldots$
$1, 1, 2, 4, 7, 13, \ldots, T_n, \ldots$	$1, 2, 3, \ldots, n, \ldots$

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Conjecture

If $\{a_j\}$ (eventually) satisfies a linear homogeneous recurrence relation with constants coefficients, then so does $\mathrm{Pf}(\{a_j\})$.

A Reduction with Row and Column Operations

Theorem

If A is skew-symmetric and we obtain B from A by

1. *adding a multiple of row i to row j and*
2. *adding the same multiple of column i to column j*

then B is skew-symmetric and $\mathrm{Pf}(A) = \mathrm{Pf}(B)$.

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$$\mathrm{Pf}(\{F_j\})_3 = \mathrm{Pf}\begin{pmatrix} 0 & 1 & 1 & 2 & 3 & 5 \\ -1 & 0 & 1 & 1 & 2 & 3 \\ -1 & -1 & 0 & 1 & 1 & 2 \\ -2 & -1 & -1 & 0 & 1 & 1 \\ -3 & -2 & -1 & -1 & 0 & 1 \\ -5 & -3 & -2 & -1 & -1 & 0 \end{pmatrix}$$

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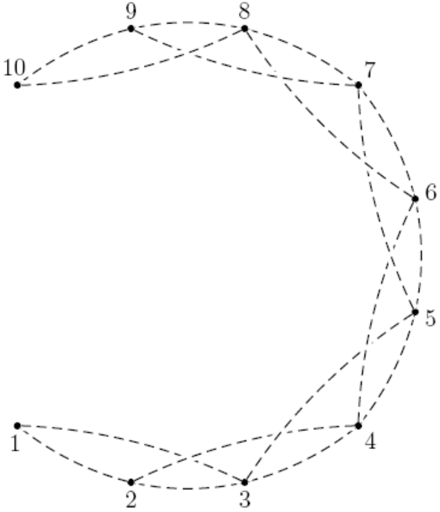
Conclusion: We can assume $a_j = 0$ for all large j.

The Claw Graph



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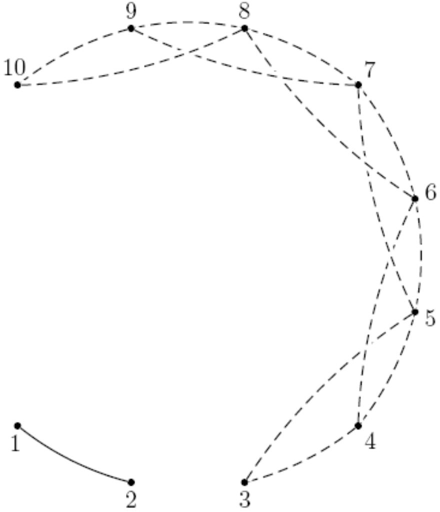

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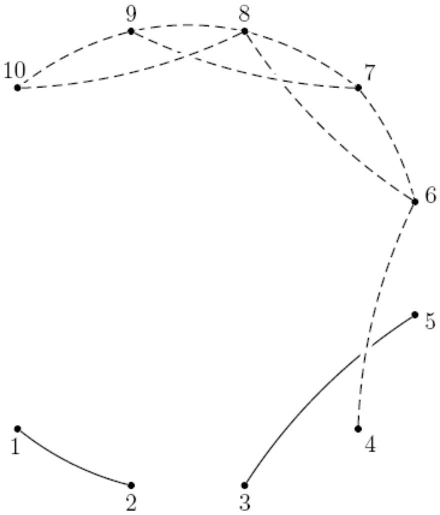

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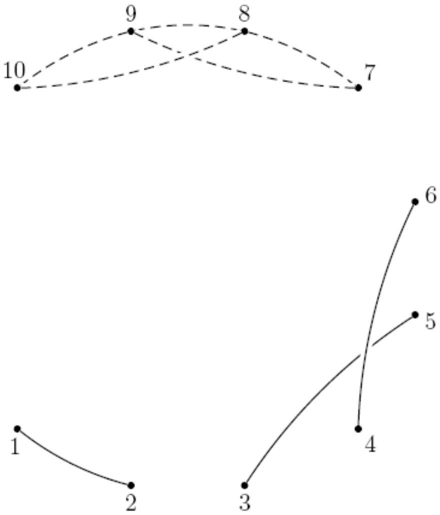

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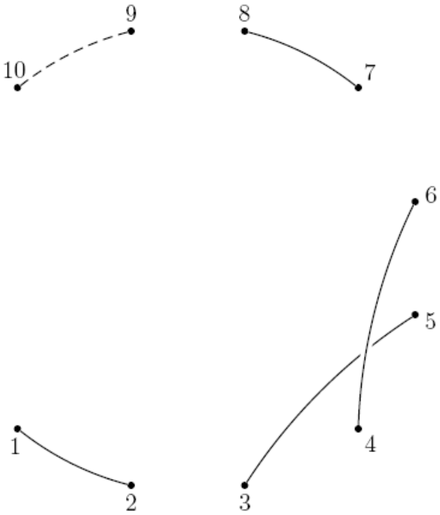

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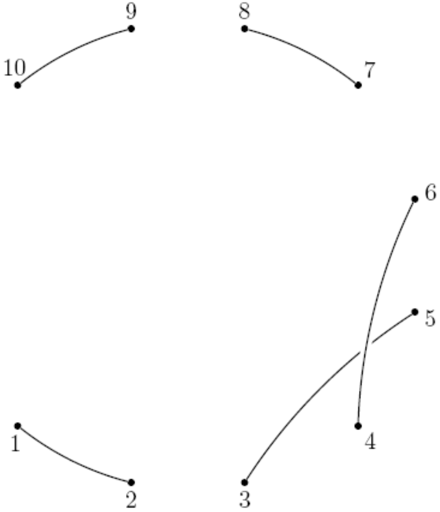

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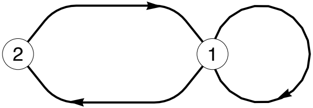

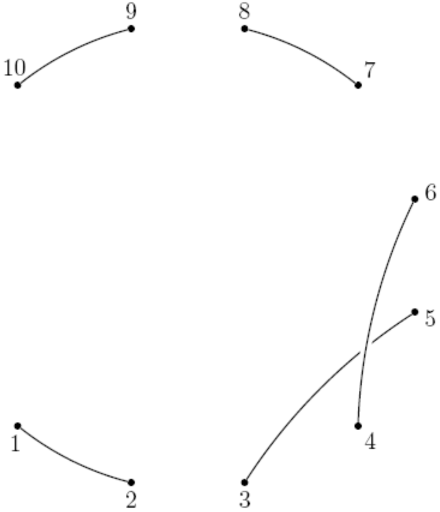

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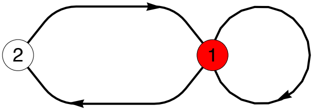

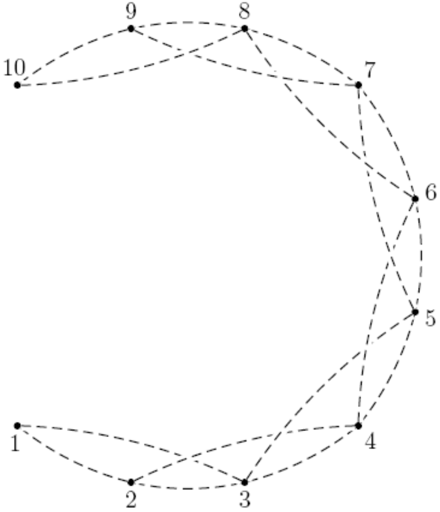

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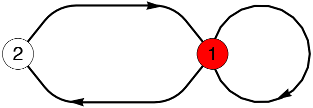

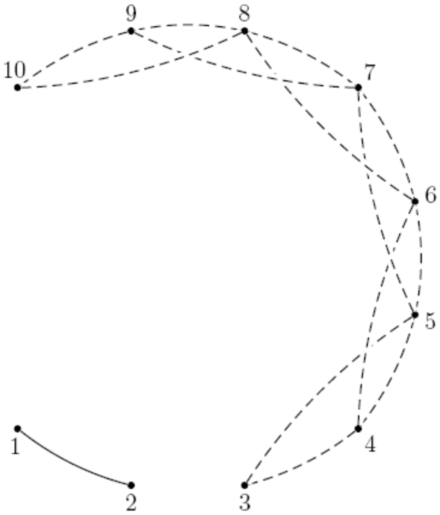

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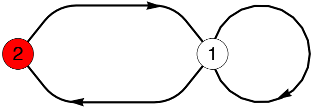

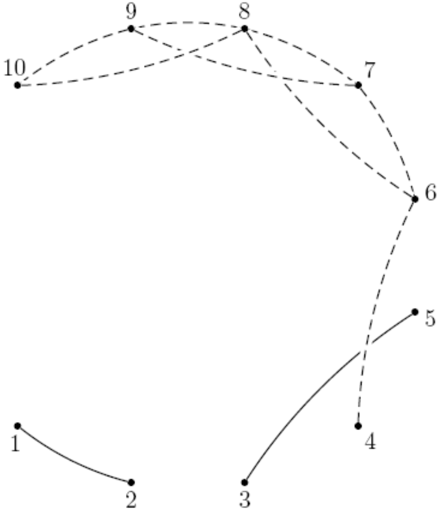

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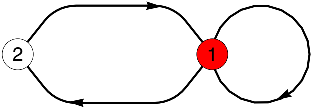

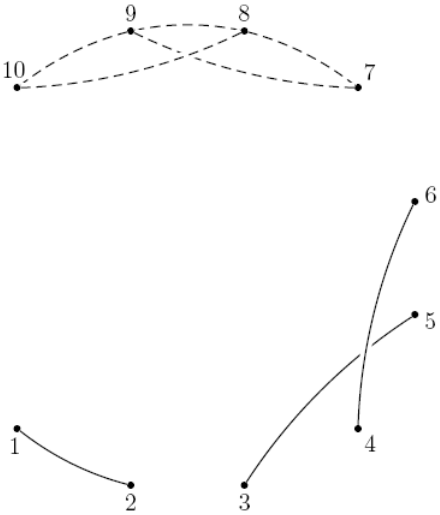

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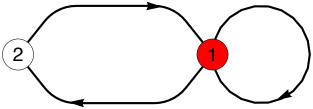

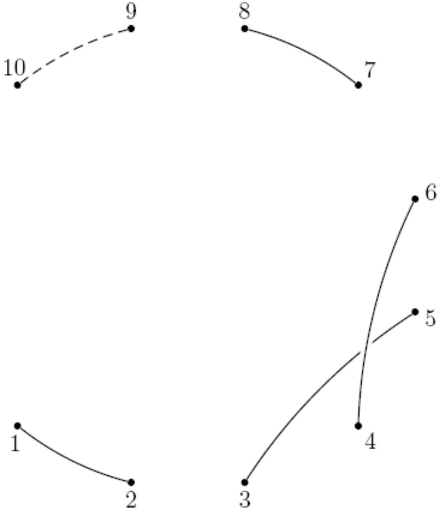

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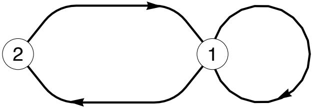

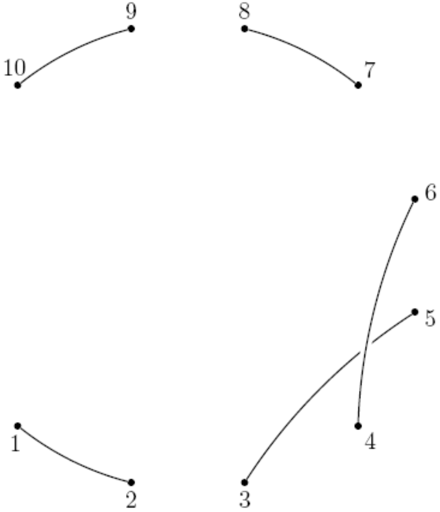

The State Digraph

Definition

The k-claw on $2n$ vertices is the graph with vertices $1, 2, \ldots, 2n$ in which vertices i and j are adjacent whenever $0 < |i - j| \leq k$.

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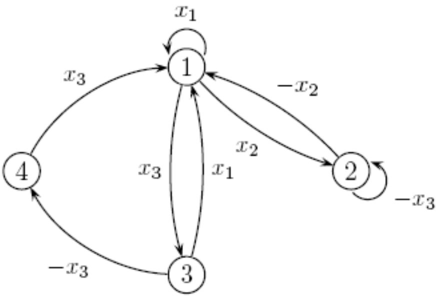
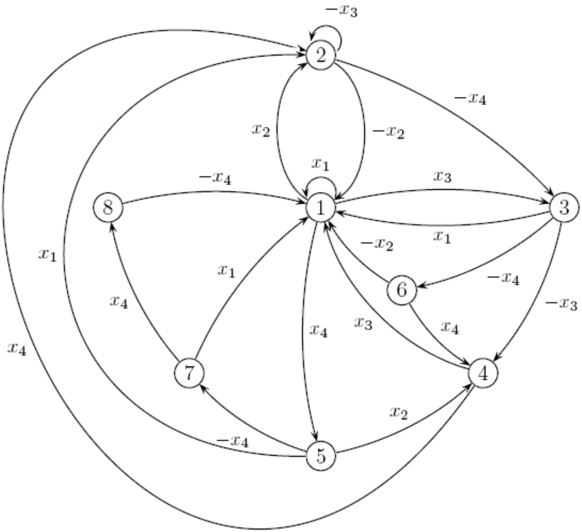
These perfect matchings are in bijection with paths in a certain digraph, called the state digraph.

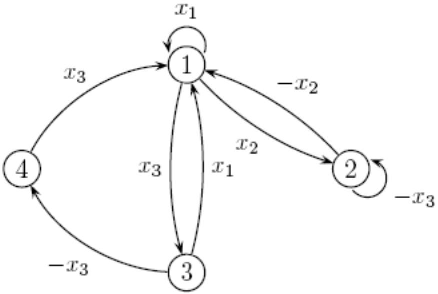
States in the 3-Claw



States in the 4-Claw



The Adjacency Matrix for the 3-Claw



$$A_3 = \begin{pmatrix} x_1 & x_2 & x_3 & 0 \\ -x_2 & -x_3 & 0 & 0 \\ x_1 & 0 & 0 & -x_3 \\ x_3 & 0 & 0 & 0 \end{pmatrix}$$

Deus Ex Transfer Matrix Method

Theorem

$$\sum_{n=1}^{\infty} \mathrm{Pf}(x_1, x_2, \ldots, x_k, 0, \ldots)_n t^n = \frac{\det(I - tA_k; 1, 1)}{\det(I - tA_k)}$$

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Corollary

If $\{a_j\}$ *satisfies a linear recurrence relation with constant coefficients then so does* $\mathrm{Pf}(\{a_j\})$*.*

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Bonus: We can find the recurrence relation in terms of $x_1, \ldots, x_k$.

The End

Thank You!