Pattern Avoiding Fishburn Permutations and Ascent Sequences

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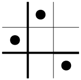
Fishburn Permutations

π is a *Fishburn permutation* whenever π avoids 

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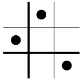
π is a *Fishburn permutation* whenever π avoids 

in words: whenever π has no subsequence $\pi_j\pi_{j+1}\pi_k$ with

- $\pi_{j+1} > \pi_j$ and
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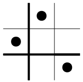
π is a *Fishburn permutation* whenever π avoids 

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Fishburn permutations are counted by the Fishburn numbers, whose generating function is

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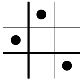
π is a *Fishburn permutation* whenever π avoids 

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A022493
1
1
2
5
15
53
217
1014
5335
31240
201608
1422074

Ascent Sequences

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ascent sequences are also counted by the Fishburn numbers

The BCDK Bijection

Fact: if $\pi \in S_n$ is a Fishburn permutation and ρ is the permutation we get by removing n from π then ρ is also a Fishburn permutation.

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Consequence: we can build Fishburn permutations inserting the entries $1, 2, \ldots, n$ one at a time.

Idea: to obtain an ascent sequence from π, record the positions of insertion in the construction of π.

The BCDK Bijection: an Example

Permutation	Ascent Sequence
1	0

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${}^{0}1^{1}$	0

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1 2	01

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${}^{0}1^{1}$	0
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${}^{0}3\ 1^{1}5\ 2^{2}4^{3}$	01021

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<u>Notation</u>

$F_n(\sigma_1, \ldots, \sigma_k)$	$A_n(\sigma_1, \ldots, \sigma_k)$
Fishburn permutations which avoid $\sigma_1, \ldots, \sigma_k$	ascent sequences which avoid $\sigma_1, \ldots, \sigma_k$

$F_n(3412)$ and $A_n(201)$

Conjecture (Gil and Weiner)

$|F_n(3412)| = |A_n(201)|$ *for all* $n \geq 0$.

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Theorem (E)

For all $n \geq 0$, *the BCDK map is a bijection between* $F_n(3412)$ *and* $A_n(201)$.

Only 123 and 321 are interesting in S_3

Goal: investigate $|F_n(\sigma_1, \ldots, \sigma_k)|$ for various $\sigma_1, \ldots, \sigma_k$.

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Theorem (E)

If any of $\sigma_1, \ldots, \sigma_k$ is one of $132, 231, 213, 312$ then $F_n(\sigma_1, \ldots, \sigma_k) = S_n(231, \sigma_1, \ldots, \sigma_k)$.

$F_n(123)$ and $A_n(012)$

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For all $n \geq 0$, the generating function $\sum_{\pi \in F_n(123)} q^{\mathrm{inv}(\pi)} t^{\mathrm{lrmax}(\pi)}$ is given by

$$t \sum_{s=0}^{n-2} q^{s^2+s-ns+\binom{n}{2}} \begin{bmatrix} n-2 \\ s \end{bmatrix}_q + t^2 \sum_{s=1}^{n-1} q^{s^2-ns+\binom{n}{2}} \begin{bmatrix} n-2 \\ s-1 \end{bmatrix}_q .$$

$F_n(123, \sigma)$

Theorem (E)

If $\sigma \in F_k(123)$ *and* $\sigma \neq k\ k-1 \cdots 21$ *then*

$$|F_n(123, \sigma)| = \sum_{j=0}^{k-2} \binom{n-1}{j}.$$

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If $\sigma \in F_k(123)$ *and* $\sigma \neq k\ k-1 \cdots 21$ *then*

$$|F_n(123, \sigma)| = \sum_{j=0}^{k-2} \binom{n-1}{j}.$$

Observation: σ could contain the Fishburn pattern and still restrict $F_n(123)$.

$F_n(123, \sigma)$

Definition

σ restrictive: avoids 123, contains the Fishburn pattern, and avoids both 2413 and 3412.

σ unrestrictive: avoids 123, contains the Fishburn pattern, and contains at least one of 2413 and 3412.

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If σ is unrestrictive then $F_n(123, \sigma) = F_n(123)$ for all $n \geq 0$.

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Theorem (E)

If $k \geq 3$ and $\sigma \in S_k$ is restrictive then for all $n \geq 0$,

$$|F_n(123, \sigma)| = \sum_{j=0}^{k-1} \binom{n-1}{j}.$$

$F_n(321, 1423)$

n	0	1	2	3	4	5	6	7	8	9
$\lvert F_n(321, 1423) \rvert$	1	1	2	4	8	15	27	47	80	134

$F_n(321, 1423)$

Theorem (E)

The generating tree for the Fishburn permutations which avoid 321 *and* 1423 *is given by the following.*

Root: $(2x)$

Rules:

$(1) \to (2z)$

$(2x) \to (1), (3)$

$(2y) \to (2y), (2z)$

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$(3) \to (2y), (1), (3)$

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Theorem (E)

For all $n \geq 0$ we have

$$|F_n(321, 1423)| = F_{n+2} - n - 1.$$

Here $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

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The generating tree for the Fishburn permutations which avoid 321 *and* 3124 *is given by the following.*

Root: (2)

Rules:

$$(1a) \to (2)$$

$$(1b) \to (1b)$$

$$(k) \to \underbrace{(1b), \ldots, (1b)}_{k-2}, (1a), (k+1)$$

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$F_n(321, 2143)$

Theorem (E)

For all $n \geq 0$, we have

$$\sum_{\pi \in F_n(321,2143)} t^{\mathrm{lrmax}(\pi)} = t(t+1)^{n-1}.$$

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Theorem (E)

For all $n \geq 0$, we have

$$\sum_{\pi \in F_n(321,2143)} t^{\mathrm{lrmax}(\pi)} = t(t+1)^{n-1}.$$

Corollary

The number of $\pi \in F_n(321, 2143)$ with exactly k left to right maxima is $\binom{n-1}{k-1}$.

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For all $n \geq 0$, we have

$$|F_n(321, 2143)| = 2^{n-1}.$$

Conjectures and Open Problems

Open Problem

Find a constructive bijection between $F_n(321, 1423)$ *or* $F_n(321, 3124)$ *and the set of binary sequences of length* $n-1$ *in which no two consecutive* 1*s have more than one* 0 *between them.*

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Conjecture (Cerbai, Claesson, Ferrari)

$A_n(201)$ *is in bijection with permutations in* S_n *which are 312-machine sortable.*

Conjectures and Open Problems

Conjecture (E)

For all $n \geq 0$,

$$|F_n(1243, 2134)| = |S_n(123, 3241)|.$$

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$$|F_n(1243, 2134)| = |S_n(123, 3241)|.$$

Conjecture (E)

For all $n \geq 1$,

$$|F_n(1324, 2143)| = (n-1)2^{n-2} + 1.$$

The Last Slide

Thank You!