The Last Small Restricted Symmetric Permutation Enumeration

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Outline

1. The central enumeration problem
2. An unexpected connection leading to a potential refinement of the problem
3. An approach via simple permutations

Background

focus on classical pattern avoidance

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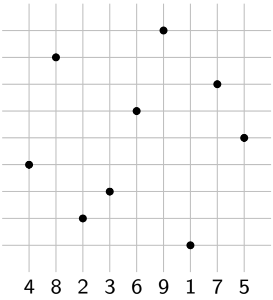
observation: the dihedral group action on permutation respects pattern containment/avoidance



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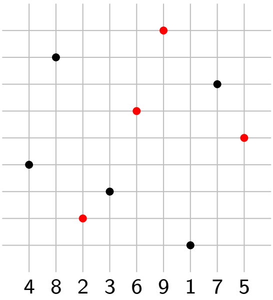

observation: the dihedral group action on permutation respects pattern containment/avoidance

π contains σ if and only if π^g contains σ^g

General Problem

Given a subgroup H,

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Notation: $Av_n^H(\sigma_1, \ldots, \sigma_k)$

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$$|Av^{90}_{4n}(1324)| = (n+1)2^{n-1}$$

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Exercises

$$|Av^{90}_{4n+2}(2413)| = |Av^{90}_{4n+3}(2413)| = 0$$

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Data and a Conjecture

n	0	1	2	3	4	5	6	7
$\lvert Av_{4n}^{90}(2413)\rvert$	1	1	3	13	67	381	2307	14589

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Conjecture (E,2007)

$$\lvert Av_{4n}^{90}(2413)\rvert = d_{n-1}$$

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$$1 + \sum_{n=1}^{\infty} d_{n-1} x^n = \frac{4}{3 + \sqrt{1-8x}} = \frac{1}{1 - xC(2x)}$$

Terms in the Recurrence

				1				
			1		2			
		3		6		4		
	13		26		12		16	
67		134		52		48		80

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Question

What statistic on $Av_{4n}^{90}(2413)$ *has this distribution?*

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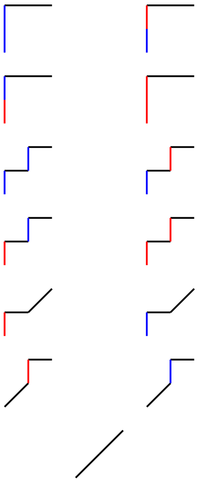

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214365
215643
216435
342165
356421
421365
435621
562143
563421
564213
621435
642135
634215

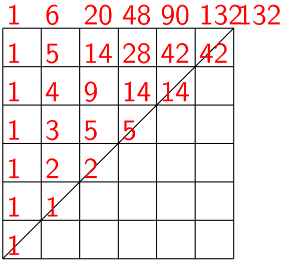
Part 2

An unexpected connection

leading to

a potential refinement of the problem

Ballot Numbers



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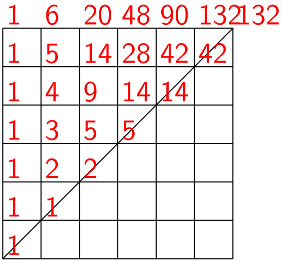

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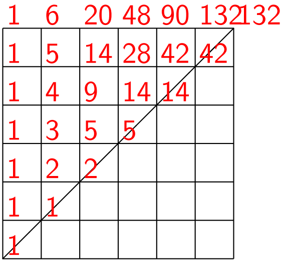

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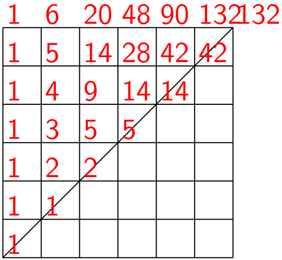
What are the generating functions for the columns?

The Ballot Column Generating Functions

$$G_m(x) = \sum_{k=m}^{\infty} b_{m,k} x^{k-m}$$



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1	6	20	48	90	132	132
1	5	14	28	42	42	
1	4	9	14	14		
1	3	5	5			
1	2	2				
1	1					
1						

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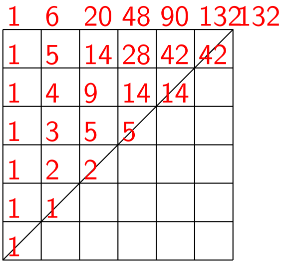

The Ballot Column Generating Functions

$$G_2(x) = \frac{2 - x}{(1-x)^3}$$

$$G_3(x) = \frac{5 - 6x + 2x^2}{(1-x)^4}$$

$$G_4(x) = \frac{14 - 28x + 20x^2 - 5x^3}{(1-x)^5}$$

$$G_5(x) = \frac{42 - 120x + 135x^2 - 70x^3 + 14x^4}{(1-x)^6}$$

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1	5	14	28	42	42	
1	4	9	14	14		
1	3	5	5			
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1						

Observations

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Theorem (E)

$$p_m(x) = \sum_{j=0}^{m-1} (-1)^j \frac{1}{m} \binom{2m}{m-j-1} \binom{m+j-1}{j} x^j$$

The Borel Triangle

				1				
			2		1			
		5		6		2		
	14		28		20		5	
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Part 3

An approach via simple permutations

Inflation

For any $\pi \in Av^{90}_{4n}$ and any nonempty permutations $\sigma_1, \ldots, \sigma_n$,

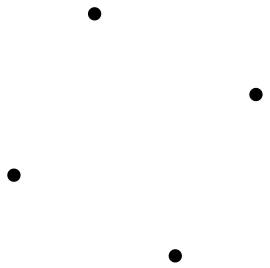
$$\pi[\sigma_1, \ldots, \sigma_n]$$

is the permutation obtained by

replacing the entries of π

with $\sigma_1, \ldots, \sigma_n$ and their rotations.

Inflation Example: 2413[132]



Replace left dot with 132

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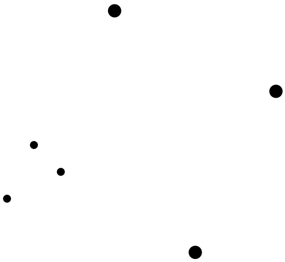

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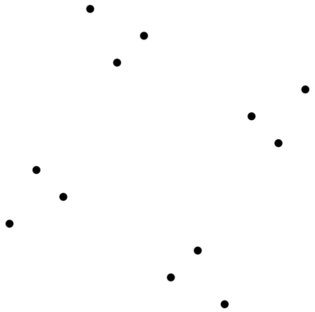

Replace images of left dot with images of 132

Simple Elements of $Av_{4n}^{90}(2413)$

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n	1	2	3	4	5	6	7	8
# simples in $Av^{90}_{4n}(2413)$	1	1	3	10	38	154	654	2871

Simple Elements of $Av_{4n}^{90}(2413)$

Conjecture (E)

The number of simple permutations in $Av_{4n}^{90}(2413)$ is the number of dissections of a convex $(n+2)$-gon into triangles and quadrilaterals by nonintersecting diagonals (A001002).

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Theorem (E)

This conjecture implies our conjectured enumeration of $Av_{4n}^{90}(2413)$.

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The End

Thank You!