A Weight-Preserving Bijection Between Schröder Paths and Schröder Permutations

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Abstract

In 1993 Bonin, Shapiro, and Simion showed that the Schröder numbers count certain kinds of lattice paths; these paths are now called Schröder paths. In 1995 West showed that the Schröder numbers also count permutations which avoid the patterns 4231 and 4132. Using some technical machinery, Barcucci, Del Lungo, Pergola, and Pinzani showed in 1999 that a certain q-analog of the Schröder numbers, called the Schröder polynomial, is the generating function for a statistic called the area statistic on Schröder paths and is also the generating function for the inversion number on permutations which avoid 4231 and 4132. In this paper we give a constructive bijection from Schröder paths to permutations which avoid 4231 and 4132 that takes the area statistic on Schröder paths to the inversion number on permutations which avoid 4231 and 4132.

Keywords: Schröder paths, Schröder permutations, Schröder polynomials, Catalan polynomials, pattern-avoiding permutations, lattice paths, inversion number

1. Introduction

The Catalan numbers $C_0, C_1, \ldots$ are given in closed form by $C_n = \frac{1}{n+1}\binom{2n}{n}$, but they may also be defined by the recurrence

$$C_0 = 1,$$
$$C_n = \sum_{k=1}^{n} C_{k-1}C_{n-k} \qquad (n \geq 1).$$

It is well known that C_n is the number of lattice paths from $(0,0)$ to (n,n) using only NORTH $(0,1)$ and EAST $(1,0)$ steps which never pass below the line $y = x$. We refer to these paths as Dyck paths. It is also well known that C_n is the number of permutations of $1,2,\ldots,n$ which avoid the pattern 312. (See section 2 for definitions.) In [5] Carlitz and Riordan introduced a q-analog $C_n(q)$ of C_n which satisfies

$$C_0(q) = 1,$$
$$C_n(q) = \sum_{k=1}^{n} C_{k-1}(q)C_{n-k}(q)q^{k-1} \qquad (n \geq 1).$$

Observe that $C_n(1) = C_n$ for $n \geq 0$. In [6] Fürlinger and Hofbauer showed that $C_n(q)$ is the generating function for the area statistic on Dyck paths and for the inversion number on 312-avoiding permutations. In [1] Bandlow and Killpatrick provided a combinatorial proof of this result by giving a constructive bijection from Dyck paths to 312-avoiding permutations which takes the area statistic to the inversion number.

In [8] Garsia and Haiman generalized $C_n(q)$ by introducing a polynomial $C_n(q,t)$ such that $C_n(1,1) = C_n$ and $C_n(1,q) = C_n(q,1) = C_n(q)$; this polynomial is now called the q,t-Catalan polynomial. Garsia and Haiman conjectured that $C_n(q,t)$ is the Hilbert series of the diagonal harmonic alternates and showed that it is the coefficient of the elementary symmetric function e_n in the symmetric polynomial $DH_n(x;q,t)$, the conjectured Frobenius characteristic of the module of diagonal harmonic polynomials. It is immediate from Garsia and Haiman's definition that $C_n(q,t) = C_n(t,q)$. Moreover, Haglund [9] has found a statistic on Dyck paths, called the t-statistic, such that $C_n(q,t)$ is the generating function for the area and t-statistics on Dyck paths. One important open problem is to find an involution on Dyck paths which takes the area statistic to the t-statistic and the t-statistic to the area statistic.

Closely related to the Catalan numbers are the Schröder numbers $R_0, R_1, \ldots$, which are defined by the recurrence

$$R_0 = 1,$$
$$R_{n+1} = R_n + \sum_{k=1}^{n+1} R_{k-1}R_{n+1-k} \qquad (n \geq 0).$$

(No simple closed form for R_n is known.) In [3] Bonin, Shapiro, and Simion showed that R_n is the number of lattice paths from $(0,0)$ to (n,n) using only NORTH $(0,1)$, EAST $(1,0)$, and DIAGONAL $(1,1)$ steps which never pass below the line $y = x$. We refer to these paths as Schröder paths. In [13] West showed that R_n is also the number of permutations of $1,2,\ldots,n,n+1$ which avoid the patterns 4231 and 4132. We refer to these permutations as Schröder permutations. (For other families of pattern-avoiding permutations counted by the Schröder numbers, see [10].) The q-analogs of the Schröder numbers, called the Schröder polynomials, are defined by the recurrence

$$S_0(q) = 1,$$

$$S_{n+1}(q) = S_n(q) + \sum_{k=1}^{n+1} S_{k-1}(q) S_{n+1-k}(q) q^k \qquad (n \geq 0).$$

Observe $S_n(1) = R_n$ for $n \geq 0$. In [2] Barcucci, Del Lungo, Pergola, and Pinzani used some technical machinery to show that $S_n(q)$ is the generating function for a generalized area statistic on Schröder paths and for the inversion number on Schröder permutations.

The Schröder numbers and Catalan numbers are related by

$$R_n = \sum_{k=0}^{n} \binom{2n-k}{k} C_{n-k} \qquad (n \geq 0).$$

Moreover, observe that every Dyck path is a Schröder path and that every permutation which avoids 312 also avoids 4231 and 4132. So if we view $S_n(q)$ as a sum over Schröder paths then we obtain (a multiple of) $C_n(q)$ by restricting the sum to Dyck paths. Similarly, if we view $S_n(q)$ as a sum over Schröder permutations then we obtain (a multiple of) $C_{n+1}(q)$ by restricting the sum to 312-avoiding permutations.

Our main goal is to find a symmetric q,t-Schröder polynomial which generalizes $C_n(q,t)$. Recently Egge, Haglund, Killpatrick, and Kremer have found a q,t-Schröder polynomial which generalizes $C_n(q,t)$, but it is not known whether this polynomial is symmetric. In this paper we advance on our goal by extending properties of $C_n(q)$ to $S_n(q)$. Specifically, we generalize the work of Bandlow and Killpatrick by giving a constructive bijection from Schröder paths to Schröder permutations which takes the area statistic to the inversion number.

In section 2 we give the necessary definitions and background for this paper. In section 3 we construct a bijection between Schröder paths and Schröder permutations and prove that this bijection maps the area statistic to the inversion number. In section 4 we discuss some open problems related to this work.

2. Schröder Numbers and Polynomials

The Schröder numbers $R_0, R_1, \ldots$ are defined by the recurrence

$$R_0 = 1,$$

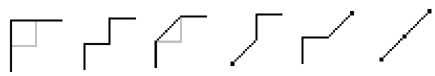
$$R_{n+1} = R_n + \sum_{k=1}^{n+1} R_{k-1} R_{n+1-k} \qquad (n \geq 0) \tag{2.1}$$

and form the sequence

$$\{R_n\}_{n=0}^{\infty} = \{1, 2, 6, 22, \ldots\}.$$

A *Schröder path* is a lattice path in $\mathbb{Z}^2$ from $(0,0)$ to (n,n) consisting of steps in the $(0,1)$ direction (NORTH steps), the $(1,0)$ direction (EAST steps) and the $(1,1)$

direction (DIAGONAL steps) such that there are no points (x,y) on the path for which $x > y$. In other words, a Schröder path is a path from $(0,0)$ to (n,n) consisting only of NORTH, EAST and DIAGONAL steps that never goes below the diagonal. We write P_n to denote the set of Schröder paths from $(0,0)$ to (n,n). For example, P_2 consists of the paths illustrated below.



The Schröder number R_n is known to count Schröder paths from $(0,0)$ to (n,n), thus $R_2 = 6$. The *length* of a Schröder path is the number of NORTH and DIAGONAL steps in the path, thus a Schröder path $\pi \in P_n$ has length n. Schröder paths which contain no DIAGONAL steps are called *Dyck* paths and the number of such paths is given by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. The Schröder numbers are related to the Catalan numbers by

$$R_n = \sum_{k=0}^{n} \binom{2n-k}{k} C_{n-k} \qquad (n \geq 0).$$

This relation can be explained by counting Schröder paths according to how many DIAGONAL steps they contain. Specifically, for $0 \leq k \leq n$ let D_k denote the set of Schröder paths from $(0,0)$ to (n,n) which contain exactly k DIAGONAL steps. Clearly $R_n = \sum_{k=0}^{n} |D_k|$, so it is sufficient to show $|D_k| = \binom{2n-k}{k} C_{n-k}$. To do this, observe that if a Schröder path has k DIAGONAL steps then it has a total of $2n-k$ steps. To form a Schröder path with k DIAGONAL steps, first choose which of the $2n-k$ steps will be diagonal. This can be done in $\binom{2n-k}{k}$ ways. Then fill in the remaining steps with a Dyck path of length $n-k$, which can be done in C_{n-k} ways. It follows that $|D_k| = \binom{2n-k}{k} C_{n-k}$, as desired.

Both the Schröder numbers and the Catalan numbers have many other combinatorial interpretations. See [12, Exercise 6.19, p. 219] for an extensive list of combinatorial interpretations of the Catalan numbers. See [12, Exercise 6.39, p. 239] for an extensive list of combinatorial interpretations of the Schröder numbers.

The recurrence (2.1) satisfied by the Schröder number can also be visualized using the Schröder paths. For $k \geq 2$, let

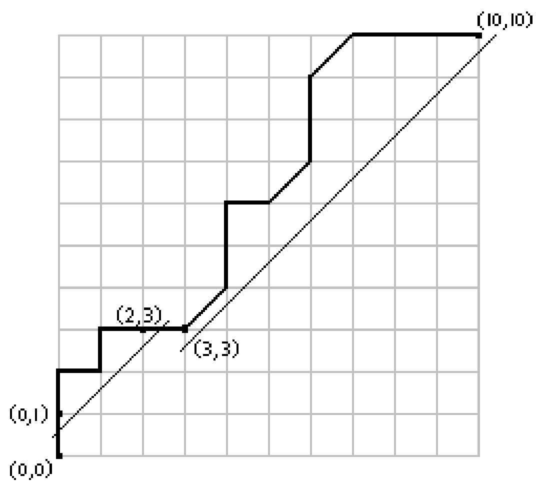
$$A_k = \{\text{Schröder paths from } (0,0) \text{ to } (n+1,n+1) \text{ that first touch } y = x \text{ at } (k,k)\}.$$

In other words, for $k \geq 2$, the set A_k consists of those Schröder paths for which k is the smallest positive integer such that (k,k) is a point on the path. In addition, let A_1 be the set of paths that start with a NORTH step and then an EAST step and let A_1' be the set of paths that start with a DIAGONAL step. Then clearly $R_{n+1} = |A_1'| + \sum_{k=1}^{n+1} |A_k|$. It remains to show that $|A_k| = R_{k-1} R_{n+1-k}$ and $|A_1'| = R_n$.

If a path first touches the diagonal at (k,k) then it must go from $(0,1)$ to $(k-1,k)$ without touching the diagonal points $(1,1)$, $(2,2)$, $\ldots$, $(k-1,k-1)$. The number of such paths is R_{k-1}. Once the path touches (k,k) it must then continue to $(n+1,n+1)$ without going below or to the right of the diagonal. The number of such paths is R_{n+1-k}. Thus for $k \geq 2$ we have $|A_k| = R_{k-1} R_{n+1-k}$. Paths in A_1 start with a NORTH step followed by an EAST step and then can take any valid path from $(1,1)$ to $(n+1,n+1)$. The number of such paths is R_n. (Note: $R_n = R_0R_n$ since $R_0 = 1$.) Paths in A'_1 start with a DIAGONAL step and then can take any valid path from $(1,1)$ to $(n+1,n+1)$. Again there are R_n ways to do this so $|A'_1| = R_n$. Therefore,

$$R_{n+1} = |A'_1| + \sum_{k=1}^{n+1} |A_k| = R_n + \sum_{k=1}^{n+1} R_{k-1}R_{n+1-k}.$$

For example, if $n = 9$ and $k = 3$, then any path in A_3 must go from $(0,0)$ to $(0,1)$, then take some path from $(0,1)$ to $(2,3)$ without touching $(1,1)$ or $(2,2)$. Since the chosen path is in A_3, it must then go from $(2,3)$ to $(3,3)$ and then it can take any valid Schröder path from $(3,3)$ to $(10,10)$. One example of such a path is illustrated below.



The Schröder numbers also count certain kinds of pattern-avoiding permutations. A *4132-avoiding* permutation $\pi \in S_n$ is a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ containing no subsequence $\pi_i\pi_j\pi_k\pi_l$ with $i < j < k < l$ such that $\pi_i > \pi_k > \pi_l > \pi_j$. That is, we say π avoids 4132 whenever it contains no subsequence whose elements are in the same relative order as 4132. A *4231-avoiding* permutation is defined similarly. A *Schröder permutation* is a permutation that is both 4132- and 4231-avoiding. The Schröder number R_n counts the Schröder permutations in S_{n+1} [2]. (See [10] for other sets of pattern-avoiding permutations counted by the Schröder numbers.)

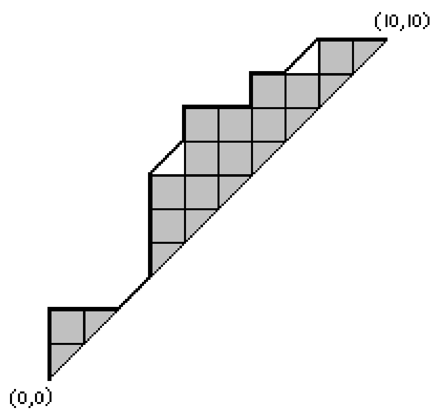
A *statistic* on a permutation, lattice path, or other combinatorial object counts some property about that object. The *inversion* statistic (or inversion number) of a permutation $\sigma \in S_n$ is defined by

$$inv(\sigma) = \sum_{\substack{1\le i<j\le n \\ \sigma_i>\sigma_j}} 1.$$

For example, if $\sigma = 743216598$, then $inv(\sigma) = 14$ since each of the pairs 21, 31, 41, 71, 32, 42, 72, 43, 73, 74, 65, 75, 76, and 98 contributes 1 to the sum. The generating function for the inversion statistic on S_n is given by

$$\sum_{\sigma \in S_n} q^{inv(\sigma)}.$$

In addition to defining statistics on permutations, we can define statistics on Schröder paths. Given a Schröder path $\pi \in P_n$, the *area* statistic, $a(\pi)$, is the number of full squares and upper half-squares that lie below the path and completely above the diagonal. For example, for the Schröder path shown below the squares counted by the area statistic are shaded



The generating function for the area statistic on Schröder paths is given by

$$\sum_{\pi \in P_n} q^{a(\pi)} = S_n(q),$$

and is known as the Schröder polynomial [3]. Specializing $q = 1$ in the Schröder polynomial gives the usual Schröder number R_n while restricting the sum to paths using only NORTH and EAST steps (Dyck paths) gives a multiple of the q-Catalan polynomial. Barcucci, Del Lungo, Pergola, and Pinzani [2] showed that

$$S_{n+1}(q) = S_n(q) + \sum_{k=1}^{n+1} S_{k-1}(q) S_{n+1-k}(q) q^k \qquad (n \geq 0). \tag{2.2}$$

To visualize this recurrence, use notation similar to our explanation of the recurrence for the Schröder numbers. That is, for $k \geq 2$, let A_k denote the set of Schröder paths for which k is the smallest positive integer such that (k,k) is a point on the path. In addition, let A_1 denote the set of paths that start with a NORTH step and then an EAST step and let A'_1 denote the set of paths that start with a DIAGONAL step. For $1 \leq k \leq n+1$, let

$$A_k(q) = \sum_{\pi \in A_k} q^{a(\pi)}$$

and

$$A'_1(q) = \sum_{\pi \in A'_1} q^{a(\pi)}.$$

Clearly,

$$S_{n+1}(q) = A'_1(q) + \sum_{k=1}^{n+1} A_k(q).$$

Then to understand the Schröder recurrence in (2.2), it is necessary to understand why

$$A_k(q) = S_{k-1}(q)S_{n+1-k}(q)q^k$$

and why

$$A'_1(q) = S_n(q).$$

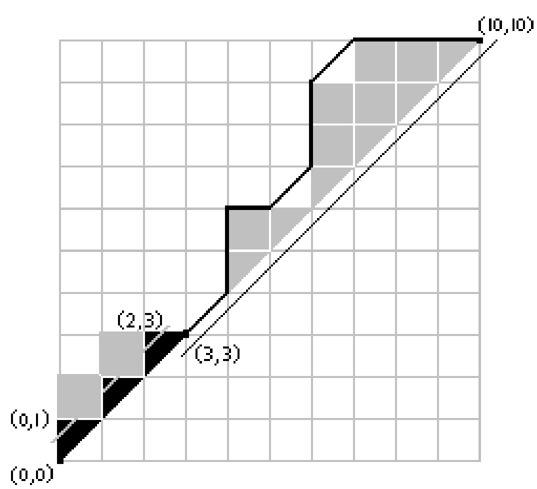
Since a path in A_k first touches the diagonal at (k,k) it must go from $(0,1)$ to $(k-1,k)$ without touching the diagonal points $(1,1)$, $(2,2)$, $\ldots$, $(k-1,k-1)$. The number of such paths is R_k and the sum of their weights is $S_k(q)$. To these paths we must add the k half-squares just above the diagonal from $(0,0)$ to (k,k). Thus the parts of the paths from $(0,0)$ to (k,k) in A_k give us a weight of $q^k S_k(q)$. From (k,k) the paths must continue on to $(n+1,n+1)$ without going below the diagonal. These paths have weight $S_{n+1-k}(q)$, giving a total weight for paths in A_k of

$$A_k(q) = q^k S_{k-1}(q)S_{n+1-k}(q).$$

Any path in A'_1 starts with a DIAGONAL step, which has a weight of zero, and then continues on from $(1,1)$ to $(n+1,n+1)$, so the total weight of these paths is just $S_n(q)$. Thus

$$S_{n+1}(q) = S_n(q) + \sum_{k=1}^{n+1} q^k S_{k-1}(q)S_{n+1-k}(q).$$

Using the previous example of a path in A_3, the additional 3 half squares of weight q^3 are shaded in black in the diagram below.



3. A Bijection Between Schröder Paths and Schröder Permutations

Before stating and proving our main theorem, we will describe a well-defined method for writing any permutation $\sigma \in S_n$ as a product of adjacent transpositions which will prove useful.

Let $\sigma \in S_n$ and let s_i denote the transposition that interchanges the number in position i with the number in position $i+1$ when applied to σ. Write σ as a product of adjacent transpositions s_i by first determining a specific sequence of adjacent transpositions which, when applied to σ, will give the identity permutation. Then σ can be represented by the inverse of this sequence of transpositions.

To determine the specific sequence of adjacent transpositions, suppose n is in position i in σ. Then $s_{n-1}s_{n-2}\cdots s_{i+1}s_i$ (applied right to left) moves the n to position n and leaves the relative order of the numbers 1 through $n-1$ unchanged. Now locate $n-1$ in the resulting permutation. Suppose $n-1$ is in position j. Then the sequence $s_{n-2}s_{n-3}\cdots s_{j+1}s_j$ moves the $n-1$ to position $n-1$. Continuing in this manner will give the identity permutation. Then σ can be represented as the inverse of this sequence of transpositions. Since $s_i^2 = id$, it follows that $s_i^{-1} = s_i$ so the inverse of this sequence of transpositions is the same sequence written in reverse order. Thus σ is represented by a product of adjacent transpositions s_i whose subscripts form a series of increasing subsequences, i.e., $\sigma = \sigma_1\sigma_2\cdots\sigma_j$ with $j \le n$ such that each σ_i is a product of adjacent transpositions whose subscripts are strictly increasing. In this representation, j is the minimum number of such subsequences.

For example, let

$$\sigma = 2\ 3\ 1\ 6\ 8\ 7\ 9\ 5\ 10\ 4.$$

Then s_9 moves the 10 to the last position, giving

$$s_9(\sigma) = 2\ 3\ 1\ 6\ 8\ 7\ 9\ 5\ 4\ 10.$$

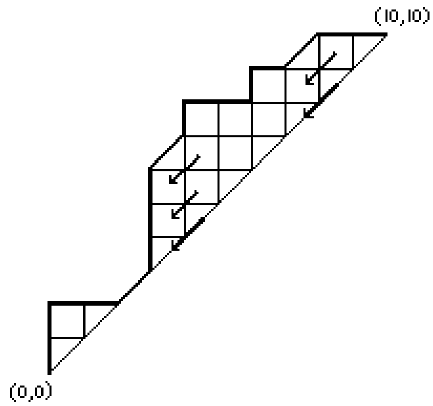
Next s_8s_7 moves the 9 to the 9th position, $s_7s_6s_5$ moves the 8 to the 8th position, s_6s_5 moves the 7 to the 7th position, s_5s_4 moves the 6 to the 6th position, s_4 moves the 5 to the 5th position, the 4 is already in the 4th position, s_2 moves the 3 to the 3rd position, and s_1 moves the 2 to the 2nd position. Then σ can be represented as the inverse of this sequence of transpositions, so

$$\sigma = s_9\ /\ s_7s_8\ /\ s_5s_6s_7\ /\ s_5s_6\ /\ s_4s_5\ /\ s_4\ /\ s_2\ /\ s_1.$$

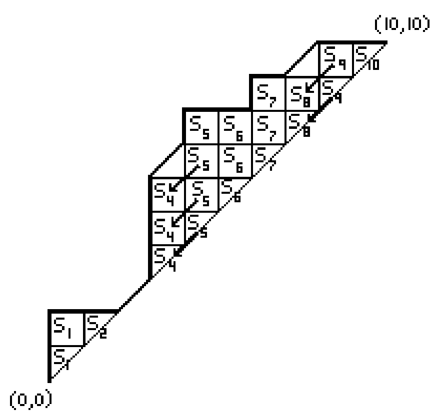
(The symbol / has been added above only as a delimiter for the sake of readability.) In this example, $\sigma = \sigma_1\sigma_2\cdots\sigma_8$ where $\sigma_1 = s_9$, $\sigma_2 = s_7s_8$, $\sigma_3 = s_5s_6s_7$, $\sigma_4 = s_5s_6$, $\sigma_5 = s_4s_5$, $\sigma_6 = s_4$, $\sigma_7 = s_2$, and $\sigma_8 = s_1$.

We now use this method of writing a permutation as a product of transpositions to describe a function f from P_n to the set of Schröder permutations in S_{n+1}. This function will turn out to be a bijection which takes the area statistic to the inversion number.

Suppose π is a path in P_n. Then for each lower half-square below the path that lies at the top of column j, draw a diagonal arrow from each square in column $j+1$ and row k to the square in column j and row $k-1$, for each $k > j$. For example, if π is the path in the picture below, then draw the diagonal arrows as illustrated.



Label shaded full squares and upper half-squares on the diagonal with an s_j if the square is in column j. Then let σ_1 be the sequence of s_i's with decreasing subscripts obtained by reading the top row of squares below the path from right to left, writing an s_j for each shaded square or upper half-square in column j. If there is a diagonal arrow at the end of row k in column $i+1$, follow the diagonal arrow down to the square in row $k-1$ and column i and continue to read the sequence of s_i's in shaded squares from right to left. When there are no more squares to the left that lie under the path or no more diagonal arrows at the end of the row, then σ_1 is complete. Follow the same process to obtain σ_2, starting with the row of shaded squares that lies below the topmost row. Filling in the example from above with the correct s_i's we obtain the following picture.



From this picture we find that $\sigma_1 = s_{10}s_9s_8s_7$, $\sigma_2 = s_9s_8s_7s_6s_5$, $\sigma_3 = s_7s_6s_5s_4$, $\sigma_4 = s_6s_5s_4$, $\sigma_5 = s_5s_4$, $\sigma_6 = s_2s_1$ and $\sigma_7 = s_1$.

For each i, let σ'_i be the sequence of s_j's in σ_i written in reverse order. In the above example, $\sigma'_1 = s_7s_8s_9s_{10}$, $\sigma'_2 = s_5s_6s_7s_8s_9$, $\sigma'_3 = s_4s_5s_6s_7$, $\sigma'_4 = s_4s_5s_6$, $\sigma'_5 = s_4s_5$, $\sigma'_6 = s_1s_2$ and $\sigma'_7 = s_1$. We define $f(\pi) \in S_{n+1}$ by writing $f(\pi) = \sigma'_1\sigma'_2\cdots\sigma'_k$. In the example above, $f(\pi) = 3\ 2\ 1\ 8\ 10\ 7\ 11\ 6\ 4\ 5\ 9$.

Lemma 1. *If π is a Schröder path then $f(\pi)$ is a Schröder permutation.*

Proof. We argue by induction on n, the length of the Schröder path. The result is immediate when $n = 1$, since 12 and 21 are both Schröder permutations.

Now assume that if π is a Schröder path of length $n-1$ then $f(\pi)$ is a Schröder permutation in S_n. Let $\hat{\pi}$ be a Schröder path of length n. If there are no squares or upper half-squares under the path in the top row, then f maps $\hat{\pi}$ to a permutation with $n+1$ in the $(n+1)$st position. In this case, it is enough to check that the permutation in positions 1 through n is a Schröder permutation. By induction, the path from $(0,0)$ to $(n-1,n-1)$ maps to a permutation in S_n that is a Schröder permutation, thus adding $n+1$ to the end of the permutation still gives a Schröder permutation.

Suppose there exist squares or upper half-squares under the Schröder path $\hat{\pi}$ in row n. Let $\sigma_1 = s_n s_{n-1} \cdots s_j s_{j-1} \cdots s_i$ be the sequence of transpositions obtained when reading shaded squares from right to left starting with the top row, as in the definition of f. Removing the squares associated with these transpositions from the Schröder path leaves a Schröder path of length $n-1$ which by induction maps to a Schröder permutation in S_n. Let α denote this permutation in S_n. It remains to check that $s_i \cdots s_{j-1} s_j \cdots s_{n-1} s_n(\alpha(\mathbf{n+1}))$ is 4132 and 4231-avoiding, i.e. a Schröder permutation.

Suppose the permutation α ends in **(n-k+1)(n-k+2)**$\cdots$ **(n-1) n** and **n-k** is in position i in α with $i \leq n-k-1$. Then the element in position $n-k$ of α is a number less than $n-k$. With this in mind, we will make use of the following result.

Lemma 2. *The permutation $s_i \cdots s_{j-1} s_j \cdots s_{n-1} s_n(\alpha(\mathbf{n+1}))$ can fail to be 4132 or 4231 avoiding only if* **n+1** *moves two or more positions to the left of* **n-k**.

Assuming for the moment that Lemma 2 holds, it is enough to show that **n+1** moves at most one position to the left of **n-k**, for **n-k** defined as above.

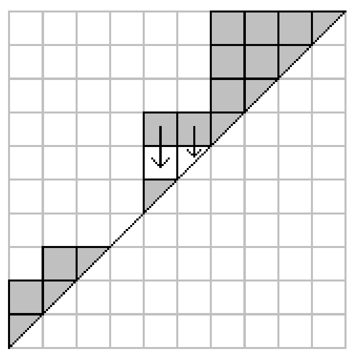
Recall that $f(\hat{\pi}) = s_i \cdots s_{j-1} s_j \cdots s_{n-1} s_n(\alpha(\mathbf{n+1}))$ for some i. If $i \geq n-k-1$, then $s_i \cdots s_n$ can move **n+1** at most one position to the left of **n-k**, since **n-k** is in position l in α with $l \leq n-k-1$. Thus the resulting permutation is both 4132 and 4231 avoiding and so $f(\hat{\pi})$ is a Schröder permutation.

Suppose $i < n-k-1$ and let **n-k** be in position l in α. Since **(n-k+1)(n-k+2)**$\cdots$**(n-1)n** remain fixed in α, then $s_{n-k+1} \ldots s_{n-1} s_n$ correspond to upper half-squares on the diagonal in the Schröder path and s_{n-k} corresponds to the full square in column $n-k$ and row $n-k+1$ in the Schröder path. The remaining $s_i \ldots s_{n-k-1}$ also must correspond to full squares in the Schröder path. However, each of these full squares must have either a full square or an upper half-square below it, since α corresponds to a Schröder path. Thus $i \geq l$ and so **n+1** can move at most one position past $n-k$ in $f(\hat{\pi})$. Therefore $f(\hat{\pi})$ is a Schröder permutation. ∎

We now give a proof of Lemma 2.

Proof. (of Lemma 2) Suppose $\ldots \mathbf{n+1} \ldots \mathbf{i} \ldots \mathbf{k} \ldots \mathbf{j} \ldots$ forms a 4132 or 4231 pattern for some $k < n-k$. If **n-k** is to the left of i, then $\ldots \mathbf{n-k} \ldots \mathbf{i} \ldots \mathbf{k} \ldots \mathbf{j} \ldots$ would have formed a 4132 or 4231 pattern in α, but α is a Schröder permutation so it avoids these patterns. If **n-k** is between **i** and **j** then $\ldots \mathbf{n+1} \ldots \mathbf{i} \ldots \mathbf{n-k} \ldots \mathbf{j} \ldots$ is a 4132 or a 4231 pattern. In this case, **n+1** is two or more positions to the left of **n-k**. If **n-k** is to the right of **j**, then $\ldots \mathbf{n+1} \ldots \mathbf{i} \ldots \mathbf{n-k} \ldots \mathbf{l} \ldots$ is a 4132 or a 4231 pattern, where l is the element in position $n-k$ so $l < n-k$. Again in this case, **n+1** is two or more positions to the left of **n-k**. ∎

To show f is a bijection, we describe its inverse map. To do this, suppose we are given a Schröder permutation σ. Use the method described at the beginning of this section to write σ as a product of transpositions, obtaining $\alpha = \sigma'_1\sigma'_2\cdots\sigma'_k$. Recall that each σ'_i is a subsequence of adjacent transpositions with increasing subscripts. For each i, if σ'_i has length l and ends with s_m, then shade in the squares of $\mathbb{Z}^2$ in the mth row and in columns m through $m-l+1$. For example, if the given Schröder permutation is $\sigma = 3\ 4\ 2\ 1\ 8\ 6\ 11\ 10\ 9\ 5\ 7$ then $\alpha = s_7s_8s_9s_{10}s_7s_8s_9s_7s_8s_5s_6s_7s_5s_2s_3s_1s_2s_1$ and the shaded squares are as in the following picture.



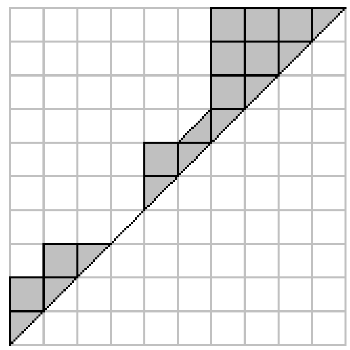
To obtain a Schröder path from the diagram, we slide certain of the shaded squares down, using the following procedure.

(1) Find the right-most column containing an unshaded square or upper half-square with a shaded square above. Choose the highest such square or half-square in the column.

(2) Shift all of the shaded squares which are both above and weakly to the left of the chosen unshaded square or half-square down by one square. If the area chosen is a half-square, then the chosen half-square becomes shaded and the top shaded square in its column becomes a shaded lower half-square.

(3) Repeat steps 1 and 2 until there are no unshaded squares or upper half-squares below a shaded square or half-square.

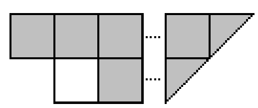
Once the sliding procedure is complete, we define $g(\sigma)$ to be the Schröder path atop the resulting shaded region. In our example we obtain the path illustrated below.



The following result concerning the sliding procedure above will be useful.

Lemma 3. *Suppose σ is a Schröder permutation. In step 1 of the sliding procedure above, the unshaded area selected will always be an upper half-square on the diagonal.*

Proof. Arguing by contradiction, suppose at some stage the unshaded area chosen is a square. Then the part of our diagram surrounding this square looks as follows.



If the transpositions corresponding to the top row of this picture are $s_{m-l} \dots s_m$ then the transpositions corresponding to the second row are $s_{m-l+k} \dots s_{m-1}$ for some k such that $0 < k < l-1$. These two rows of transpositions will produce a pattern of type $m+1\ a\ m\ b$ in σ, where $a \leq m-1$ and $b \leq m-1$. If $a < b$ then this is a 4132 pattern. If $b < a$ then this is a 4231 pattern. In either case, we have a contradiction. ∎

Lemma 4. *Let f and g be the maps described above. If π is a Schröder path then $g(f(\pi)) = \pi$. If σ is a Schröder permutation then $f(g(\sigma)) = \sigma$. In particular, $g = f^{-1}$.*

Proof. Suppose σ is a Schröder permutation, and that $\sigma = \sigma_1 \dots \sigma_k$ when written as a product of transpositions as previously described, in which each σ_i is a product of transpositions with increasing subscripts. By the first part of the construction of $g(\sigma)$, the string σ_i may be written in the shaded squares and upper half-squares in row i of the diagram obtained before applying the sliding procedure. By Lemma 3 and the construction of f, these strings are never broken under the sliding procedure involved in the construction of g. The result follows. ∎

Lemma 5. *Suppose σ is a permutation with k inversions. Then when σ is written as a product of adjacent transpositions as described then σ has exactly k terms in the product. In other words, every transposition in the product representation of σ corresponds to an inversion in σ.*

Proof. Suppose $\sigma = \sigma_1\sigma_2\cdots\sigma_k$ when written as a product of transpositions in the manner described, in which each σ_i is a product of transpositions with increasing subscripts. Suppose $\sigma_i = s_j s_{j+1}\cdots s_l$. Then this sequence of transpositions repeatedly interchanges the position of $l+1$ with the element to the left of $l+1$. By construction the element to the left of $l+1$ is always less than $l+1$, so each transposition introduces exactly one inversion. ∎

Theorem 1. *The function f is a bijection from Schröder paths to Schröder permutations that maps the area statistic to the inversion number.*

Proof. It is immediate from Lemma 4 that f is a bijection. It follows from Lemma 5 and the construction of f that f maps a Schröder path with area statistic k to a Schröder permutation with inversion number k. ∎

4. Open Problems

As yet, no generalization of the q,t-Catalan polynomial to a q,t-Schröder polynomial $S_n(q,t)$ such that $S_n(q,1) = S_n(1,q) = S_n(q)$ and $S_n(q,t) = S_n(t,q)$ is known. However, Egge, Haglund, Killpatrick, and Kremer have recently found a t-statistic that is equidistributed with the area statistic on Schröder paths. In other words, if $S_n(q,t)$ is the generating function for the area statistic and this new t-statistic on Schröder paths from $(0,0)$ to (n,n) then $S_n(q,1) = S_n(1,q) = S_n(q)$. It is not known whether $S_n(q,t) = S_n(t,q)$. The problem of finding an involution on Dyck paths from $(0,0)$ to (n,n) which reverses the area and t-statistics also remains open.

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