Legendre-Stirling Permutations*

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Abstract

We first give a combinatorial interpretation of Everitt, Littlejohn, and Wellman's Legendre-Stirling numbers of the first kind. We then give a combinatorial interpretation of the coefficients of the polynomial $(1-x)^{3k+1}\sum_{n=0}^{\infty} \left\{\!\!\left\{ {n+k \atop n} \right\}\!\!\right\} x^n$ analogous to that of the Eulerian numbers, where $\left\{\!\!\left\{ {n \atop k} \right\}\!\!\right\}$ are Everitt, Littlejohn, and Wellman's Legendre-Stirling numbers of the second kind. Finally we use a result of Bender to show that the limiting distribution of these coefficients as n approaches infinity is the normal distribution.

Keywords: descent, Stirling number, Legendre-Stirling number

1 Introduction

Following Knuth [6], let $\left[{n \atop k} \right]$ and $\left\{ {n \atop k} \right\}$ denote the (unsigned) Stirling numbers of the first and second kinds, respectively, which may be defined by the initial conditions

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = \delta_{n,0}, \qquad \begin{bmatrix} 0 \\ k \end{bmatrix} = \delta_{k,0} \tag{1}$$

and

$$\begin{Bmatrix} n \\ 0 \end{Bmatrix} = \delta_{n,0}, \qquad \begin{Bmatrix} 0 \\ k \end{Bmatrix} = \delta_{k,0} \tag{2}$$

and recurrence relations

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}, \qquad (n,k \in \mathbb{Z}), \tag{3}$$

and

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} + k\begin{Bmatrix} n-1 \\ k \end{Bmatrix}, \qquad (n,k \in \mathbb{Z}). \tag{4}$$

It is well known that $\left[{n \atop k} \right]$ and $\left\{ {n \atop k} \right\}$ have a variety of interesting algebraic properties; for instance,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{Bmatrix} -k \\ -n \end{Bmatrix}, \qquad (n,k \in \mathbb{Z}), \tag{5}$$

$$\sum_{k=1}^{n} (-1)^{j+k} \begin{bmatrix} i \\ k \end{bmatrix} \begin{Bmatrix} k \\ j \end{Bmatrix} = \delta_{i,j}, \qquad (1 \le i,j \le n), \tag{6}$$

and

$$\sum_{k=1}^{n} (-1)^{j+k} \begin{Bmatrix} i \\ k \end{Bmatrix} \begin{bmatrix} k \\ j \end{bmatrix} = \delta_{i,j}, \qquad (1 \le i,j \le n). \tag{7}$$

*2000 Mathematics Subject Classification: Primary 05A15; Secondary 05A18

The Stirling numbers of each kind also have combinatorial interpretations: for $n \geq 1$ and $k \geq 1$ the quantity $\left[n \atop k \right]$ is the number of permutations of $[n]$ with exactly k cycles, while $\left\{ n \atop k \right\}$ is the number of partitions of $[n]$ with exactly k blocks.

Recently Everitt, Littlejohn, and Wellman introduced [4] the Legendre-Stirling numbers of the second kind, which may be defined by the initial conditions

$$\left\{\!\!\left\{ n \atop 0 \right\}\!\!\right\} = \delta_{n,0}, \qquad \left\{\!\!\left\{ 0 \atop k \right\}\!\!\right\} = \delta_{k,0} \tag{8}$$

and recurrence relation

$$\left\{\!\!\left\{ n \atop k \right\}\!\!\right\} = \left\{\!\!\left\{ n-1 \atop k-1 \right\}\!\!\right\} + k(k+1) \left\{\!\!\left\{ n-1 \atop k \right\}\!\!\right\}, \qquad (n, k \in \mathbb{Z}). \tag{9}$$

It is not difficult to show that when $n \geq 1$ we have

$$x^n = \sum_{j=0}^{n} \left\{\!\!\left\{ n \atop j \right\}\!\!\right\} \langle x \rangle_j, \tag{10}$$

where $\langle x \rangle_j = x(x-2)(x-6)\cdots(x-(j-1)j)$. These numbers first arose in the study of a certain differential operator related to Legendre polynomials, but Andrews and Littlejohn [1] have given them the following combinatorial interpretation. For each $n \geq 1$, let $[n]_2$ denote the set $\{1_1, 1_2, 2_1, 2_2, \ldots, n_1, n_2\}$, which consists of two distinguishable copies of each positive integer from 1 to n. By a *Legendre-Stirling set partition* of $[n]_2$ into k blocks we mean an ordinary set partition of $[n]_2$ into $k+1$ blocks for which the following hold.

1. One block, called the *zero block*, is distinguished, but all other blocks are indistinguishable.
2. The zero block may be empty, but all other blocks are nonempty.
3. The zero block may not contain both copies of any number.
4. Each nonzero block contains both copies of the smallest number it contains, but does not contain both copies of any other number.

Then Andrews and Littlejohn have shown [1] that the number of Legendre-Stirling set partitions of $[n]_2$ into k blocks is $\left\{\!\!\left\{ n \atop k \right\}\!\!\right\}$, by showing that these two quantities satisfy the same initial conditions and recurrence relation.

In this paper we prove Legendre-Stirling analogues of a variety of results concerning Stirling numbers of the first and second kinds. In section 2 we give a recursive definition of the Legendre-Stirling numbers of the first kind, which we denote by $\left[\!\left[n \atop k \right]\!\right]$. We then prove analogues of (5), (6), and (7) for the Legendre-Stirling numbers, and we give a combinatorial interpretation of $\left[\!\left[n \atop k \right]\!\right]$ in terms of pairs of permutations of $[n]$ with k cycles. In sections 3 and 4 we turn our attention to $f_k(n) = \left\{\!\!\left\{ n+k \atop n \right\}\!\!\right\}$ and $g_k(n) = \left[\!\left[n-1 \atop n-k-1 \right]\!\right]$, which are the kth northwest to southeast diagonals of the second and first Legendre-Stirling triangles, respectively. We show that $f_k(n)$ is a polynomial of degree $3k$ in n with $f_k(0) = f_k(-1) = \cdots = f_k(-k-1) = 0$; we show that similar results hold for $g_k(n)$ by showing that $g_k(n) = (-1)^k f_k(-n)$. These results, together with standard facts concerning rational generating functions, imply that there exist integers $B_{k,j}$ such that

$$\sum_{n=0}^{\infty} f_k(n) x^n = \frac{\sum_{j=1}^{2k-1} B_{k,j} x^j}{(1-x)^{3k+1}}.$$

We give two combinatorial interpretations of $B_{k,j}$, the second of which involves descents in a certain family of permutations, which we call Legendre-Stirling permutations. The results in these two sections are analogues of results of Gessel and Stanley [5] concerning the Stirling numbers. In section 5 we first show that for any $k \geq 1$ the sequence $\{B_{k,j}\}_{j=1}^{2k-1}$ is unimodal. We then turn our attention to the random variable X_k, which is the number of descents in a uniformly chosen Legendre-Stirling permutation. We show that

$$E[X_k] = \frac{6k-1}{5}, \qquad (k \geq 1),$$

and

$$Var[X_k] = \frac{(k-1)(108k+99)}{525k-175}, \qquad (k \geq 1),$$

and we combine these results with a theorem of Bender to show that $\left\{\frac{X_k - E[X_k]}{\sqrt{Var[X_k]}}\right\}_{k=1}^{\infty}$ converges in distribution to the standard normal variable. These results are analogues of results of Bóna [3] concerning the Stirling numbers.

2 Legendre-Stirling Numbers of the First Kind

Andrews and Littlejohn [1] define the Legendre-Stirling numbers of the first kind $\left[\!\left[{n \atop k} \right]\!\right]$ via

$$\langle x \rangle_n = \sum_{j=0}^{n} (-1)^{n+j} \left[\!\left[{n \atop j} \right]\!\right] x^j, \tag{11}$$

where $\langle x \rangle_j = x(x-2)(x-6)\cdots(x-(j-1)j)$ as above, but they say nothing else about these quantities. In this section we give a recursive definition of $\left[\!\left[{n \atop k} \right]\!\right]$, which we use to prove analogues of (5), (6), and (7) and to give a combinatorial interpretation of $\left[\!\left[{n \atop k} \right]\!\right]$.

Definition 2.1 *For all $n, k \in \mathbb{Z}$ we write $\left[\!\left[{n \atop k} \right]\!\right]$ to denote the* (signless) Legendre-Stirling numbers of the first kind*, which are given by the initial conditions*

$$\left[\!\left[{n \atop 0} \right]\!\right] = \delta_{n,0}, \qquad \left[\!\left[{0 \atop k} \right]\!\right] = \delta_{k,0}, \tag{12}$$

and recurrence relation

$$\left[\!\left[{n \atop k} \right]\!\right] = \left[\!\left[{n-1 \atop k-1} \right]\!\right] + n(n-1) \left[\!\left[{n-1 \atop k} \right]\!\right], \qquad (n, k \in \mathbb{Z}). \tag{13}$$

It is not difficult to show that (11) and Definition 2.1 are equivalent for $n, k \geq 1$, so we turn our attention to an analogue of (5).

Theorem 2.2 *For all $n, k \in \mathbb{Z}$,*

$$\left\{\!\!\left\{ {-k \atop -n} \right\}\!\!\right\} = (-1)^{k+n} \left[\!\left[{n-1 \atop k-1} \right]\!\right]. \tag{14}$$

Proof. The Legendre-Stirling numbers of the second kind are uniquely determined by (8) and (9), so it is sufficient to show that the numbers $L(n,k) = (-1)^{k+n} \left[\!\left[{-k-1 \atop -n-1} \right]\!\right]$ also satisfy (8) and (9).

To prove $L(n,k)$ satisfies the left equation in (8), first note that $L(1,0) = 0$ by (12). Now if $n \neq 1$ then set $n = 0$ and $k = n$ in (13) and use (12) to find that $L(n,0) = \delta_{n,0}$. The proof that $L(n,k)$ satisfies the right equation in (8) is similar. To prove that $L(n,k)$ satisfies (9), note that if $n \neq 0$ and $k \neq 0$ then we have

$$\begin{aligned} L(n-1,k-1) &= (-1)^{n+k} \left(-k(-k-1) \left[\!\left[{-k-1 \atop -n} \right]\!\right] + \left[\!\left[{-k-1 \atop -n-1} \right]\!\right] \right) \\ &= -(-1)^{n+k-1} k(k+1) \left[\!\left[{-k-1 \atop -n} \right]\!\right] + (-1)^{n+k} \left[\!\left[{-k-1 \atop -n-1} \right]\!\right] \\ &= -k(k+1)L(n-1,k) + L(n,k), \end{aligned}$$

and the result follows. □

The following analogues of (6) and (7) are clear from the relationship between (10) and (11), but for completeness we give a proof using the recursive definitions of $\left[\!\left[{n \atop k} \right]\!\right]$ and $\left\{\!\!\left\{ {n \atop k} \right\}\!\!\right\}$.

Theorem 2.3 *If $n \geq 1$ then for all i, j with $1 \leq i, j \leq n$ we have*

$$\sum_{k=1}^{n} (-1)^{k+j} \left[\!\!\left[{i \atop k} \right]\!\!\right] \left\{\!\!\left\{ {k \atop j} \right\}\!\!\right\} = \delta_{i,j} \tag{15}$$

and

$$\sum_{k=1}^{n} (-1)^{k+j} \left\{\!\!\left\{ {i \atop k} \right\}\!\!\right\} \left[\!\!\left[{k \atop j} \right]\!\!\right] = \delta_{i,j}. \tag{16}$$

Proof. To prove (15), first note that if $i < n$ then $\left[\!\left[{i \atop n} \right]\!\right] = 0$, and the result follows by induction on n. On the other hand, if $i = n$ then by (13), (9), and induction on n we have

$$\begin{aligned}
\sum_{k=1}^{n} (-1)^{k+j} \left[\!\!\left[{n \atop k} \right]\!\!\right] \left\{\!\!\left\{ {k \atop j} \right\}\!\!\right\} &= \sum_{k=1}^{n} (-1)^{k+j} \left(n(n-1) \left[\!\!\left[{n-1 \atop k} \right]\!\!\right] + \left[\!\!\left[{n-1 \atop k-1} \right]\!\!\right] \right) \left\{\!\!\left\{ {k \atop j} \right\}\!\!\right\} \\
&= n(n-1) \sum_{k=1}^{n-1} (-1)^{k+j} \left[\!\!\left[{n-1 \atop k} \right]\!\!\right] \left\{\!\!\left\{ {k \atop j} \right\}\!\!\right\} + \sum_{k=1}^{n-1} (-1)^{k+j} \left[\!\!\left[{n-1 \atop k-1} \right]\!\!\right] \left\{\!\!\left\{ {k \atop j} \right\}\!\!\right\} \\
&= \delta_{j,n-1} n(n-1) + \sum_{k=1}^{n-1} (-1)^{k+j} \left[\!\!\left[{n-1 \atop k-1} \right]\!\!\right] \left(\left\{\!\!\left\{ {k-1 \atop j-1} \right\}\!\!\right\} + j(j+1) \left\{\!\!\left\{ {k-1 \atop j} \right\}\!\!\right\} \right) \\
&= \delta_{j,n-1} n(n-1) + \delta_{n,j} - \delta_{j,n-1} j(j+1) \\
&= \delta_{n,j}.
\end{aligned}$$

The proof of (16) is similar to the proof of (15). □

The Stirling numbers of the first kind count permutations of $[n]$ with k cycles; we conclude this section with an analogous interpretation of the Legendre-Stirling numbers of the first kind. Here the *cycle maxima* of a given permutation are the numbers which are largest in their cycles. For example, if $\pi = (4, 6, 1)(9, 2, 3)(7, 8)$ is a permutation in S_{10}, written in cycle notation, then its cycle maxima are $5, 6, 8, 9$, and 10.

Definition 2.4 *A* Legendre-Stirling permutation pair *of length n is an ordered pair (π_1, π_2) with $\pi_1 \in S_{n+1}$ and $\pi_2 \in S_n$ for which the following hold.*

1. *π_1 has one more cycle than π_2.*
2. *The cycle maxima of π_1 which are less than $n + 1$ are exactly the cycle maxima of π_2.*

Theorem 2.5 *For all $n \geq 0$ and all k with $0 \leq k \leq n$, the number of Legendre-Stirling permutation pairs (π_1, π_2) of length n in which π_2 has exactly k cycles is $\left[\!\left[{n \atop k} \right]\!\right]$.*

Proof. Let $a_{n,k}$ denote the number of Legendre-Stirling permutation pairs (π_1, π_2) of length n in which π_2 has exactly k cycles. It is clear that $a_{n,0} = \delta_{n,0}$ and $a_{0,k} = \delta_{k,0}$, so in view of (13) it is sufficient to show that if $n > 0$ and $k > 0$ then $a_{n,k} = n(n-1)a_{n-1,k} + a_{n-1,k-1}$. To do this, first note that by condition 3 of Definition 2.4, if (π_1, π_2) is a Legendre-Stirling permutation pair of length n then 1 is a fixed point in π_1 if and only if it is a fixed point in π_2. Pairs (π_1, π_2) in which 1 is a fixed point are in bijection with pairs (σ_1, σ_2) of length $n-1$ in which σ_2 has $k-1$ cycles by removing the 1 from each permutation and decreasing all other entries by 1. Each pair (π_1, π_2) in which 1 is not a fixed point may be constructed uniquely by choosing a pair (σ_1, σ_2) of length $n-1$ in which σ_2 has k cycles, increasing each entry of each permutation by 1, and inserting 1 after an entry of each permutation. There are $a_{n-1,k}$ pairs (σ_1, σ_2), there are n ways to insert a new entry into σ_1, and there are $n-1$ ways to insert a new entry into σ_2. Now the result follows. □

3 Legendre-Stirling Polynomials

It is natural to arrange the Legendre-Stirling numbers of each kind in a triangle; Figures 1 and 2 show the first five rows of each of these triangles. Following Gessel and Stanley's study [5] of the Stirling numbers of each kind, in this section we give some elementary properties of the sequences parallel to the upper right sides of these triangles.

				1				
			2		1			
		12		8		1		
	144		108		20		1	
2880		2304		508		40		1

Figure 1: The First Legendre-Stirling Triangle.

				1				
			2		1			
		4		8		1		
	8		52		20		1	
16		320		292		40		1

Figure 2: The Second Legendre-Stirling Triangle.

Beginning with the second Legendre-Stirling triangle, it is not difficult to show that

$$\left\{\!\!\left\{ {n \atop n} \right\}\!\!\right\} = 1, \qquad (n \ge 1), \tag{17}$$

$$\left\{\!\!\left\{ {n+1 \atop n} \right\}\!\!\right\} = 2\binom{n+1}{3} + 2\binom{n+1}{2}, \qquad (n \ge 1), \tag{18}$$

and

$$\left\{\!\!\left\{ {n+2 \atop n} \right\}\!\!\right\} = 40\binom{n+2}{6} + 72\binom{n+2}{5} + 36\binom{n+2}{4} + 4\binom{n+2}{3}, \qquad (n \ge 1). \tag{19}$$

These formulas suggest the following result.

Theorem 3.1 *For all $k \ge 0$, the quantity $\left\{\!\!\left\{ {n+k \atop n} \right\}\!\!\right\}$ is a polynomial of degree $3k$ in n with leading coefficient $\frac{1}{3^k k!}$. We write $f_k(n)$ to denote this polynomial; then for all $k \ge 1$ and all $n \in \mathbb{Z}$ we have*

$$f_k(n) = n(n+1)f_{k-1}(n) + f_k(n-1). \tag{20}$$

Proof. The result is immediate for $k = 0$, so suppose $k \ge 1$; we argue by induction on k.

By (9) we have

$$\left\{\!\!\left\{ {n+k \atop n} \right\}\!\!\right\} - \left\{\!\!\left\{ {n-1+k \atop n-1} \right\}\!\!\right\} = n(n+1)\left\{\!\!\left\{ {n+k-1 \atop n} \right\}\!\!\right\} \tag{21}$$

for all $n \in \mathbb{Z}$. By induction this implies that the first difference sequence for $\left\{\!\!\left\{ {n+k \atop n} \right\}\!\!\right\}$ is a polynomial of degree $3k-1$ in n, so $\left\{\!\!\left\{ {n+k \atop n} \right\}\!\!\right\}$ is a polynomial of degree $3k$ in n. Let $f_k(n)$ denote this polynomial; now (20) is immediate from (21). Iterating (21) and using the left equation in (8) we find that if $n \ge 1$ then

$$f_k(n) = \sum_{j=1}^{n} j(j+1)f_{k-1}(j).$$

Since $\sum_{j=0}^{n} j^{3k-1}$ is a polynomial of degree $3k$ in n with leading coefficient $\frac{1}{3k}$, by induction the leading coefficient of $f_k(n)$ is $\frac{1}{3^k k!}$. $\Box$

Although one can use the same methods to prove an analogue of Theorem 3.1 for the first Legendre-Stirling triangle, we take a different approach.

Theorem 3.2 *For all $k \ge 0$, the quantity $\left[\!\!\left[{n-1 \atop n-k-1} \right]\!\!\right]$ is a polynomial of degree $3k$ in n with leading coefficient $\frac{1}{3^k k!}$. We write $g_k(n)$ to denote this polynomial; then for all $k \ge 1$ and all $n \in \mathbb{Z}$ we have*

$$g_k(n) = g_k(n-1) + (n-1)(n-2)g_{k-1}(n-1). \tag{22}$$

Proof. By (14) we have

$$\left[\!\left[\begin{matrix} n-1 \\ n-k-1 \end{matrix} \right]\!\right] = (-1)^k f_k(-n) \tag{23}$$

for all $k \geq 0$; now the result follows from Theorem 3.1. □

The relationship between f_k and g_k implied by (23) is worth noting, since it will be useful later on.

Corollary 3.3 *For all $k \geq 0$ we have*

$$g_k(n) = (-1)^k f_k(-n). \tag{24}$$

Proof. This is immediate from (23). □

The forms of $f_1(n)$ and $f_2(n)$ in (18) and (19) also suggest the following results concerning the roots of f_k and g_k.

Theorem 3.4 *If $k \geq 1$ then*

$$f_k(0) = f_k(-1) = \cdots = f_k(-k) = f_k(-k-1) = 0 \tag{25}$$

and

$$g_k(0) = g_k(1) = \cdots = g_k(k) = g_k(k+1) = 0. \tag{26}$$

Proof. When $k = 1$ line (25) is immediate from (18), so suppose $k > 1$; we argue by induction on k.

By the left equation in (8) we have $f_k(0) = 0$, and by (20) we have

$$f_k(n) - f_k(n-1) = n(n+1)f_{k-1}(n).$$

By induction the expression on the right is zero for $0 \leq n \leq -k$, and the result follows.

In view of (24), line (26) is immediate from (25). □

4 Legendre-Stirling Permutations

We now turn our attention to the generating functions for $f_k(n)$ and $g_k(n)$, which are given by

$$F_k(x) = \sum_{n=0}^{\infty} f_k(n)x^n \tag{27}$$

and

$$G_k(x) = \sum_{n=0}^{\infty} g_k(n)x^n. \tag{28}$$

By (26) and standard results concerning rational generating functions (see [8, Cor. 4.6], for instance), there exist integers $B_{k,j}$ such that

$$F_k(x) = \frac{\sum_{j=1}^{2k-1} B_{k,j}x^j}{(1-x)^{3k+1}}, \qquad (k \geq 1), \tag{29}$$

and

$$G_k(x) = \frac{x^{k+1}\sum_{j=1}^{2k-1} B_{k,3k-2-j}x^j}{(1-x)^{3k+1}}, \qquad (k \geq 1). \tag{30}$$

In this section we give two combinatorial interpretations of $B_{k,j}$. We begin with a recurrence relation for $F_k(x)$, which we use to obtain a recurrence relation for $B_{k,j}$.

Theorem 4.1 *We have*

$$F_0(x) = \frac{1}{1-x} \tag{31}$$

and

$$F_k(x) = \frac{x}{1-x}\frac{d^2}{dx^2}\left(xF_{k-1}(x)\right), \qquad (k \ge 1). \tag{32}$$

Moreover, we also have $B_{1,j} = 2\delta_{j,1}$ *and*

$$B_{k,j} = j(j+1)B_{k-1,j} + 2j(3k-1-j)B_{k-1,j-1} + (3k-j)(3k-1-j)B_{k-1,j-2}. \tag{33}$$

Proof. Line (31) is immediate from (17), and by (20) we have

$$\begin{aligned} F_k(x) &= \sum_{n=0}^{\infty} n(n+1)f_{k-1}(n)x^n + \sum_{n=0}^{\infty} f_k(n-1)x^n \\ &= x\frac{d^2}{dx^2}(xF_{k-1}(x)) + xF_k(x), \end{aligned}$$

from which (32) follows.

Now set $k = 1$ in (32) and use (31) to find that $F_1(x) = \frac{2x}{(1-x)^4}$; hence $B_{1,j} = 2\delta_{j,1}$, as claimed. To obtain (33), first use (29) to eliminate $F_{k-1}(x)$ on the right side of (32) and simplify the result to find that

$$\begin{aligned} F_k(x) = \frac{\sum_{j-1}^{2k-3} j(j+1)B_{k-1,j}x^j}{(1-x)^{3k-1}} &+ \frac{2(3k-2)\sum_{j-1}^{2k-3}(j+1)B_{k-1,j}x^{j+1}}{(1-x)^{3k}} \\ &+ \frac{(3k-2)(3k-1)\sum_{j-1}^{2k-3} B_{k-1,j}x^{j+2}}{(1-x)^{3k+1}}. \end{aligned}$$

Now use (29) to eliminate $F_k(x)$ and clear denominators to obtain

$$\begin{aligned} \sum_{j=1}^{2k-1} B_{k,j}x^j = (1-x)^2 \sum_{j=1}^{2k-3}(j+1)jB_{k-1,j}x^j &+ 2(1-x)(3k-2)\sum_{j=1}^{2k-3}(j+1)B_{k-1,j}x^{j+1} \\ &+ (3k-2)(3k-1)\sum_{j=1}^{2k-3} B_{k-1,j}x^{j+2}. \end{aligned}$$

Finally, equate coefficients of x^j to complete the proof. □

We have the following analogue of Theorem 4.1 for $G_k(x)$.

Theorem 4.2 *We have*

$$G_1(x) = \frac{1}{1-x} \tag{34}$$

and

$$G_k(x) = \frac{x^3}{1-x}\frac{d^2}{dx^2}\left(G_{k-1}(x)\right), \qquad (k \ge 1). \tag{35}$$

Proof. This is similar to the proof of (31) and (32), using (22). □

Since $B_{1,j} = 2\delta_{j,1}$, line (33) implies that $B_{k,j}$ is a nonnegative integer for all k. We give two combinatorial interpretations of $B_{k,j}$. The first is inspired by Riordan's interpretation [7, p. 9] of similar numbers arising in the study of the usual Stirling numbers, which he gives in terms of trapezoidal words.

Definition 4.3 *For any positive integer n, a* Legendre-Stirling word *on $2n$ letters is a word $a_1a_2\cdots a_{2n}$ such that for all j with $1 \le j \le n$, the entries a_{2j-1} and a_{2j} are distinct numbers from among $1, 2, \ldots, 3j-1$.*

Theorem 4.4 *The number of Legendre-Stirling words on $2k$ letters with exactly $j+1$ different entries is $B_{k,j}$.*

Proof. Let $b_{k,j}$ denote the number of Legendre-Stirling words on $2k$ letters with exactly $j+1$ different entries. The numbers $B_{k,j}$ are determined by (33) and the fact that $B_{1,j} = 2\delta_{j,1}$, so it is sufficient to show that $b_{k,j}$ also satisfies these conditions.

The only two Legendre-Stirling words on 2 letters are 12 and 21, so $b_{1,j} = 2\delta_{j,1}$. Now suppose $k > 1$. Every Legendre-Stirling word on $2k$ letters with exactly $j+1$ different entries may be uniquely constructed by choosing a Legendre-Stirling word on $2k-2$ letters and appending two distinct numbers a_{2k-1} and a_{2k} from among $1, 2, \ldots, 3k-1$. To ensure the resulting word has exactly $j+1$ different entries, we may start with a word with exactly $j-1$ different entries and append two numbers which do not already appear, we may start with a word with exactly j different entries and append one number which already appears and one which does not, or we may start with a word with exactly $j+1$ different entries and append two numbers which already appear. These constructions may be carried out in $(3k-j)(3k-1-j)b_{k-1,j-2}$, $2j(3k-1-j)b_{k-1,j-1}$, and $j(j+1)b_{k-1,j}$ ways, respectively, and the result follows. □

Our second interpretation of $B_{n,k}$ is inspired by similar results concerning the Eulerian numbers and the usual Stirling numbers. In particular, if $a_k(n) = n^k$ and $A_k(x) = \sum_{n=0}^{\infty} a_k(n)x^n$ then there are nonnegative integers $A_{k,j}$ such that

$$A_k(x) = \frac{\sum_{j=1}^{k} A_{k,j}x^j}{(1-x)^{k+1}}, \qquad (k \geq 1).$$

Moreover, these $A_{k,j}$ are the Eulerian numbers, so $A_{k,j}$ is the number of permutations in S_k with exactly j descents. Similarly, Gessel and Stanley [5] have shown that if $c_k(n) = \left\{ {n+k \atop n} \right\}$ and $C_k(x) = \sum_{n=0}^{\infty} c_k(n)x^n$ then there are nonnegative integers $C_{k,j}$ such that

$$C_k(x) = \frac{\sum_{j=1}^{k} C_{k,j}x^j}{(1-x)^{2k+1}}, \qquad (k \geq 1).$$

Moreover, Gessel and Stanley have given a set of permutations of a certain multiset such that $C_{k,j}$ is the number of these permutations with exactly j descents. In view of these results, we would like an interpretation of $B_{k,j}$ involving descents in a family of permutations.

Definition 4.5 *For each $n \geq 1$, let M_n denote the multiset*

$$M_n = \{1, 1, \overline{1}, 2, 2, \overline{2}, \ldots, n, n, \overline{n}\},$$

in which we have two unbarred copies of each integer j with $1 \leq j \leq n$ and one unbarred copy of each such integer. Then a Legendre-Stirling permutation π *is a permutation of M_n such that if $i < j < k$ and $\pi(i) = \pi(k)$ are both unbarred, then $\pi(j) > \pi(i)$. A* descent *in a Legendre-Stirling permutation π is a number i, $1 \leq i \leq 3n$, such that $i = 3n$ or $\pi(i) > \pi(i+1)$.*

Theorem 4.6 *The number of Legendre-Stirling permutations of M_k with exactly j descents is $B_{k,j}$.*

Proof. Let $b_{k,j}$ denote the number of Legendre-Stirling permutations of M_k with exactly j descents. As in the proof of Theorem 4.4, it is sufficient to show that $b_{k,j}$ satisfies the same recurrence and initial conditions as $B_{k,j}$.

The only two Legendre-Stirling permutations of M_1 are $\overline{1}11$ and $11\overline{1}$; each of these has one descent, so $b_{1,j} = 2\delta_{j,1}$. Now suppose $k > 1$. Every Legendre-Stirling permutation of M_k may be constructed by choosing a Legendre-Stirling permutation of M_{k-1}, inserting $\overline{k}$ between two entries, and then inserting the pair kk between two entries of this new permutation. We may ensure the resulting permutation has exactly j descents in four ways.

The first way is to choose a permutation of M_{k-1} with j descents, insert $\overline{k}$ immediately after a descent, and insert kk immediately after a descent or immediately before $\overline{k}$. In this case there are $b_{k-1,j}$ ways to choose the initial permutation, j ways to insert $\overline{k}$, and $j+1$ ways to insert kk.

The second way is to choose a permutation of M_{k-1} with $j-1$ descents, insert $\overline{k}$ immediately after a descent, and insert kk immediately after a nondescent, but not immediately to the left of $\overline{k}$. In this case there are $b_{k-1,j-1}$ ways to choose the initial permutation, $j-1$ ways to insert $\overline{k}$, and $3k-1-j$ ways to insert kk.

The third way is to choose a permutation of M_{k-1} with $j-1$ descents, insert $\overline{k}$ immediately after a nondescent, and insert kk immediately after a descent or immediately to the left of $\overline{k}$. In this case there are $b_{k-1,j-1}$ ways to choose the initial permutation, $3k-1-j$ ways to insert $\overline{k}$, and $j+1$ ways to insert kk.

The fourth way is to choose a permutation of M_{k-1} with $j-2$ descents, insert $\overline{k}$ immediately after a nondescent, and insert kk immediately after a nondescent, but not immediately to the left of $\overline{k}$. In this case there are $b_{k-1,j-2}$ ways to choose the initial permutation, $3k-j$ ways to insert $\overline{k}$, and $3k-1-j$ ways to insert kk.

Combining all of these, we find that

$$b_{k,j} = j(j+1)b_{k,j} + 2j(3k-1-j)b_{k-1,j-1} + (3k-j)(3k-1-j)b_{k-1,j-2},$$

as desired. □

We conclude this section with a bijective proof of Theorem 4.6. In particular, we give a bijective proof that

$$\sum_{n=0}^{\infty} f_k(n)x^n = \frac{\sum_{j=1}^{2k-1} b_{k,j}x^j}{(1-x)^{3k+1}}, \tag{36}$$

where $b_{k,j}$ is the number of Legendre-Stirling permutations of M_k with exactly j descents. Recall from the Introduction that we have a combinatorial interpretation of $f_k(n)$ in terms of set partitions; we now give a combinatorial interpretation of the coefficient of x^n in the expression on the right.

For any Legendre-Stirling permutation π, written in one-line notation, let the *spaces* of π be the spaces between consecutive entries of π, along with the space before the first entry and the space after the last entry. Then a *slashed Legendre-Stirling permutation* is a Legendre-Stirling permutation in which spaces may contain one or more slashes. For example, $\backslash\backslash 1\overline{2}1\backslash\overline{1}2\backslash\backslash\backslash 2$ is a slashed Legendre-Stirling permutation of M_2. For any $k, n \geq 0$, let $P_{k,n}$ denote the set of slashed Legendre-Stirling permutations of M_k with n slashes, in which every descent contains at least one slash. Then we have the following expression for the generating function for $|P_{k,n}|$.

Lemma 4.7 *For all $k \geq 1$ we have*

$$\sum_{n=0}^{\infty} |P_{k,n}|x^n = \frac{\sum_{j=1}^{2k-1} b_{k,j}x^j}{(1-x)^{3k+1}}.$$

Proof. Note that we can uniquely construct all slashed Legendre-Stirling permutations of M_k by choosing a Legendre-Stirling permutation of M_k, inserting a slash into each descent, and then inserting arbitrarily many slashes into each of the $3k+1$ spaces. Thus,

$$\begin{aligned}\sum_{n=0}^{\infty} |P_{k,n}|x^n &= \left(\sum_{j=1}^{2k-1} b_{k,j}x^j\right)\left(1+x+x^2+\cdots\right)^{3k+1} \\ &= \frac{\sum_{j=1}^{2k-1} b_{k,j}x^j}{(1-x)^{3k+1}},\end{aligned}$$

as desired. □

Bijective Proof of Theorem 4.6. In view of Lemma 4.7, it is sufficient to give a bijection between $P_{k,n}$ and the set of Legendre-Stirling set partitions of $[n+k]_2$ into n blocks. To begin, we first observe that every slashed Legendre-Stirling permutation in $P_{k,n}$ may be uniquely constructed as follows. Begin with a (possibly empty) row of slashes; these will be the slashes which do not appear between any two js in our final slashed Legendre-Stirling permutation. Now for each j, $1 \leq j \leq k$, first insert $\overline{j}$ to the left of a slash, then insert jj to the left of $\overline{j}$ or to the left of a slash, and then insert a (possibly empty) row of slashes between j and j.

To describe the image of a given slashed Legendre-Stirling permutation π under our bijection, we describe how to construct this image as we construct π. First number the slashes in our initial row of slashes $1, 2, \ldots, m$, from left to right, and begin the Legendre-Stirling partition with blocks $\{i_1, i_2\}$, where $1 \leq i \leq m$. When we insert $\overline{j}$ immediately to the left of slash r, we put copy 1 of the smallest unused number into the block whose smallest elements are r_1 and r_2. When we insert jj immediately to the left of slash s, we put copy 2 of the smallest unused number into the block whose smallest elements are s_1 and s_2. If that block also contains copy 1 of same number, then we move copy 1 of that number to the zero block. When we insert jj immediately to the left of $\overline{j}$, we put copy 2 of the smallest available number into the zero block. Finally, when we insert slashes between j and j, we number them consecutively from left to right, beginning with the smallest available number.

It is not difficult to give a recursive description of the inverse of this procedure, so this map is a bijection. □

5 The Distribution of the Number of Descents

Suppose $k \geq 1$, and let X_k denote the random variable whose value is the number of descents in a Legendre-Stirling permutation of M_k, chosen uniformly at random. Figure 3 shows the distribution of X_k when $k = 8$ in blue, along with the normal distribution with the same mean and standard deviation in red. Inspired by examples like this one, and by analogous work of Bóna [3] concerning Gessel and Stanley's Stirling permutations, in this section we prove that for each $k \geq 1$ the sequence $\{B_{k,j}\}_{j=1}^{2k-1}$ is unimodal, and that X_k approaches a normal variable as k goes to infinity.

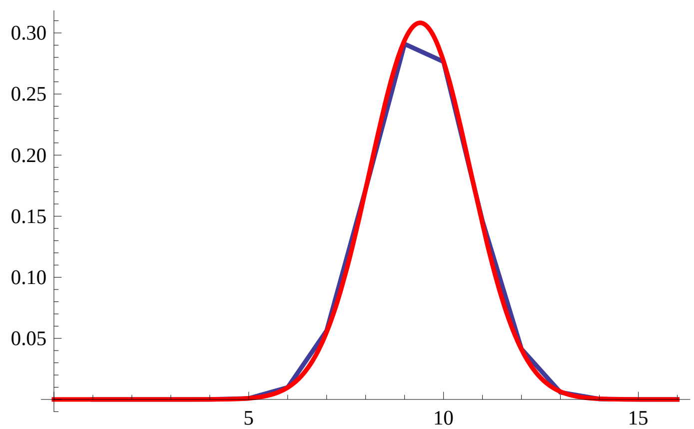

Figure 3: The distribution of X_8 and the normal distribution.

To prove $\{B_{k,j}\}_{j=1}^{2k-1}$ is unimodal, we show that the polynomial

$$B_k(x) = \sum_{j=1}^{2k-1} B_{k,j} x^j$$

has distinct, real, nonpositive roots. To do this, let $C_k(x)$ be given by

$$C_k(x) = (1-x)^{3k+2} \frac{d}{dx} \left(x(1-x)^{-1-3k} B_k(x) \right), \qquad (k \geq 1). \tag{37}$$

The table in Figure 4 gives $C_k(x)$ for $1 \leq k \leq 4$. Since $B_k(x)$ is a polynomial of degree $2k-1$, we see that

k	$C_k(x)$
1	$4x(1+x)$
2	$4x(2+23x+36x^2+9x^3)$
3	$16x(1+49x+351x^2+639x^3+324x^4+36x^5)$
4	$16x(2+335x+7056x^2+40266x^3+79470x^4+57771x^5+14400x^6+900x^7)$

Figure 4: The polynomials $C_1(x)$, $C_2(x)$, $C_3(x)$, and $C_4(x)$.

$C_k(x)$ is a polynomial of degree $2k$. Moreover, since every nonempty Legendre-Stirling permutation has at least one descent, we have $B_k(0) = 0$ for all $k \geq 1$; now it follows from (37) that $C_k(0) = 0$ for all $k \geq 1$. We can now show that the nonzero roots of $B_k(x)$ and $C_k(x)$ are negative, by showing they are intertwined.

Theorem 5.1 *For all $k \geq 1$, the polynomials $B_k(x)$ and $C_k(x)$ have distinct, real, nonpositive roots. In particular, their sequences of coefficients are unimodal.*

Proof. The result is clear for $k = 1$, since $B_1(x) = 2x$ and $C_1(x) = 4x + 4x^2$. Now suppose $k > 1$ and $B_{k-1}(x)$ and $C_{k-1}(x)$ have distinct, real, nonpositive roots; we argue by induction on k.

To see that $B_k(x)$ has distinct, real, nonpositive roots, first use (32) and the fact that $F_k(x) = \frac{B_k(x)}{(1-x)^{3k+1}}$ to show that

$$B_k(x) = x(1-x)^{3k} \frac{d}{dx} \left((1-x)^{1-3k} C_{k-1}(x)\right). \tag{38}$$

By Rolle's Theorem, $B_k(x)$ has a root strictly between each pair of consecutive roots of $C_{k-1}(x)$; including 0, this accounts for $2k-2$ of the $2k-1$ roots of $B_k(x)$. To find the last root, let $\alpha < 0$ denote the leftmost root of $C_{k-1}(x)$; by (38) we have $B_k(\alpha) = \alpha(1-\alpha)C'_{k-1}(\alpha)$. Since the degree of $C_{k-1}(x)$ is $2k-2$ we have $\lim_{x\to-\infty} C_{k-1}(x) = \infty$. Now since the roots of $C_{k-1}(x)$ are distinct we find $C'_{k-1}(\alpha) < 0$; hence $B_k(\alpha) > 0$. But the degree of $B_k(x)$ is $2k-1$, so $\lim_{x\to-\infty} B_k(x) = -\infty$, and therefore $B_k(x)$ has a root which is less than α. Now it follows that $B_k(x)$ has distinct, real, nonpositive roots.

The proof that $C_k(x)$ has distinct, real, nonpositive roots is similar, using (37).

It is well known that if a polynomial has only real, negative roots then its sequence of coefficients is unimodal; see Wilf's book [9, Prop. 4.26 and Thm. 4.27] for a proof of this fact. □

We now turn our attention to the distribution of the number of descents in a randomly chosen Legendre-Stirling permutation. To state our result precisely, we introduce some notation. For all $k \geq 1$, let $p_k(x)$ be the probability generating function for X_k, so that

$$p_k(x) = \sum_{j=1}^{2k-1} P(X_k = j)x^j,$$

where $P(X_k = j)$ is the probability that $X_k = j$. In addition, for all $k \geq 1$ let Z_k be the random variable given by $Z_k = \frac{X_k - E[X_k]}{\sqrt{Var[X_k]}}$. Here

$$E[X_k] = \sum_{j=1}^{2k-1} jP(X_k = j)$$

is the usual expected value of X_k and

$$Var[X_k] = \sum_{j=1}^{2k-1} (E(X_k) - j)^2 P(X_k = j)$$

is the usual variance of X_k. We recall that

$$Var[X_k] = E[X_k^2] - E[X_k]^2, \qquad (k \geq 1). \tag{39}$$

In our main result we prove that $\{Z_k\}_{k=1}^{\infty}$ converges in distribution to the standard normal variable; to prove this, we use the following result of Bender.

Theorem 5.2 *[2] Suppose X_k and $p_k(x)$ are as above. If all of the roots of $p_k(x)$ are real and*

$$\lim_{k\to\infty} Var[X_k] = \infty \tag{40}$$

then $\{Z_k\}_{k=1}^{\infty}$ converges in distribution to the standard normal variable.

Since $p_k(x) \left(\sum_{j=1}^{2k-1} B_{k,j}\right) = B_k(x)$, Theorem 5.1 implies all of the roots of $p_k(x)$ are real. To prove (40), we first set some additional notation. For all positive integers k and j, let $YB_{k,j}$ be the indicator variable for the event that $\overline{j}$ is not the bottom of a descent in a uniformly chosen Legendre-Stirling permutation of

M_k. Similarly, let $YL_{k,j}$ (resp. $YR_{k,j}$) be the indicator variable for the event that the left (resp. right) j is not the bottom of a descent in a uniformly chosen Legendre-Stirling permutation of M_k. Observe that

$$X_k = 3k + 1 - \sum_{j=1}^{k} (YB_{k,j} + YL_{k,j} + YR_{k,j})\,. \tag{41}$$

We prove (40) by first obtaining an explicit formula for $Var[X_k]$; as a first step, we obtain recurrences for the expected values of $YB_{k,j}$, $YL_{k,j}$, and $YR_{k,j}$.

Lemma 5.3 *Fix $k \geq 2$ and let Y be one of YB, YL, and YR. Then we have $E[Y_{k,k}] = 1$ and*

$$E[Y_{k,j}] = \frac{3k-3}{3k-1} E[Y_{k-1,j}], \qquad (1 \leq j < k). \tag{42}$$

Proof. The fact that $E[Y_{k,k}] = 1$ is immediate. For ease of exposition, suppose that $Y = YB$; the proof is identical in the other two cases. To obtain (42), first note that $E[YB_{k,j}]$ is the probability that $\overline{j}$ is not the bottom of a descent in a randomly chosen Legendre-Stirling permutation of M_k. We can obtain such a permutation by choosing a Legendre-Stirling permutation of M_{k-1} in which $\overline{j}$ is not a descent, inserting $\overline{k}$ anywhere except immediately to the left of $\overline{j}$, and then inserting kk anywhere except immediately to the left of $\overline{j}$. Thus $E[YB_{k,j}] = \frac{3k-3}{3k-2} \cdot \frac{3k-2}{3k-1} \cdot E[YB_{k-1,j}]$, and (42) follows. □

Lemma 5.3 allows us to compute $E[X_k]$, which will be useful in our computation of $Var[X_k]$.

Proposition 5.4 *For all $k \geq 1$ we have*

$$E[X_k] = \frac{6k-1}{5}. \tag{43}$$

Proof. The result is immediate for $k = 1$, so suppose $k > 1$; we argue by induction on k. Since expectation is linear, by (41), Lemma 5.3, and induction we have

$$\begin{aligned}
E[X_k] &= 3k + 1 - \sum_{j=1}^{k} (E[YB_{k,j}] + E[YL_{k,j}] + E[YR_{k,j}]) \\
&= 3k - 2 - \frac{3k-3}{3k-1} \sum_{j=1}^{k-1} (E[YB_{k-1,j}] + E[YL_{k-1,j}] + E[YR_{k-1,j}]) \\
&= 3k - 2 - \frac{3k-3}{3k-1} (3k - 2 - E[X_{k-1}]) \\
&= \frac{6k-1}{5},
\end{aligned}$$

as desired. □

The variance $Var[X_k]$ also involves expected values of products of our indicator variables, so we now find recurrence relations for these quantities.

Lemma 5.5 *Fix $k \geq 2$, let Y be one of YB, YL, and YR, and let Z be one of YB, YL, and YR. Then we have*

$$E[Y_{k,i} Z_{k,j}] = \frac{(3k-4)(3k-3)}{(3k-2)(3k-1)} E[Y_{k-1,i} Z_{k-1,j}], \qquad (1 \leq i < j < k). \tag{44}$$

Proof. This is similar to the proof of Lemma 5.3. □

We now have enough information to compute $Var[X_k]$.

Proposition 5.6 *For all $k \geq 1$ we have*

$$Var[X_k] = \frac{(k-1)(108k+99)}{525k-175}. \tag{45}$$

Proof. The result is immediate for $k = 1$, so suppose $k > 1$; we argue by induction on k. In view of (39) and (43), it is sufficient to find $E[X_k^2]$. To do this, first use (41) and linearity of expectation to obtain

$$E[X_k^2] = E\left[(3k+1)^2 - 2(3k+1)\sum_{j=1}^{k}(YB_{k,j}+YL_{k,j}+YR_{k,j}) + \left(\sum_{j=1}^{k}(YB_{k,j}+YL_{k,j}+YR_{k,j})\right)^2\right]$$
$$= (3k+1)^2 - 2(3k+1)E\left[\sum_{j=1}^{k}(YB_{k,j}+YL_{k,j}+YR_{k,j})\right] + E\left[\left(\sum_{j=1}^{k}(YB_{k,j}+YL_{k,j}+YR_{k,j})\right)^2\right].$$

Now use (41) and (43) to eliminate the expected value in the middle term on the right side, obtaining

$$E[X_k^2] = -\frac{9k^2+24k+7}{5} + E\left[\left(\sum_{j=1}^{k}(YB_{k,j}+YL_{k,j}+YR_{k,j})\right)^2\right]. \tag{46}$$

To evaluate the last term on the right, first observe that

$$\left(\sum_{j=1}^{k}(YB_{k,j}+YL_{k,j}+YR_{k,j})\right)^2 = Q_1(k) + 2Q_2(k) + Q_3(k), \tag{47}$$

where

$$Q_1(k) = \sum_{j=1}^{k}\left(YB_{k,j}^2 + YL_{k,j}^2 + YR_{k,j}^2\right),$$

$$Q_2(k) = \sum_{i,j=1}^{k}\left(YB_{k,i}YL_{k,j} + YL_{k,i}YR_{k,j} + YR_{k,i}YB_{k,j}\right),$$

and

$$Q_3(k) = \sum_{\substack{i,j=1\\ i\neq j}}^{k}\left(YB_{k,i}YB_{k,j} + YL_{k,i}YL_{k,j} + YR_{k,i}YR_{k,j}\right).$$

Since $YB_{k,j}$, $YL_{k,j}$, and $YR_{k,j}$ are always equal to 0 or 1, by (41) and (43) we have

$$E[Q_1(k)] = 3k + 1 - \frac{6k-1}{5}. \tag{48}$$

Now observe that

$$Q_2(k) = 2\sum_{i=1}^{k}(YB_{k,i}+YL_{k,i}+YR_{k,i}) - 3 + \sum_{i,j=1}^{k-1}\left(YB_{k,i}YL_{k,j} + YL_{k,i}YR_{k,j} + YR_{k,i}YB_{k,j}\right),$$

so by (41), (43), and Lemma 5.5 we have

$$E[Q_2(k)] = \frac{3}{5}(6k-1) + \frac{(3k-4)(3k-3)}{(3k-2)(3k-1)}E[Q_2(k-1)]. \tag{49}$$

Similarly, we find that

$$E[Q_3(k)] = \frac{18}{5}(k-1) + \frac{(3k-4)(3k-3)}{(3k-2)(3k-1)}E[Q_3(k-1)]. \tag{50}$$

Now combine (46), (47), (48), (49), and (50) to find that

$$E[X_k^2] = -\frac{9k^2-39k+25}{5} + \frac{(3k-4)(3k-3)}{(3k-2)(3k-1)}E[2Q_2(k-1) + Q_3(k-1)]. \tag{51}$$

To obtain an expression for $E[2Q_2(k-1)+Q_3(k-1)]$, first replace k with $k-1$ in (46) and (47) to obtain

$$\begin{aligned} Var[X_{k-1}] &= E[X_{k-1}^2] - E[X_{k-1}]^2 \\ &= -\frac{1}{5}(9k^2+6k-8) + E[Q_1(k-1)] + E[2Q_2(k-1)+Q_3(k-1)] - E[X_{k-1}]^2. \end{aligned}$$

Now replace k with $k-1$ in (48) and (43) and use the results to eliminate $E[Q_1(k-1)]$ and $E[X_{k-1}]^2$, respectively. Using induction to eliminate $Var[X_{k-1}]$ we find that

$$E[2Q_2(k-1)+Q_3(k-1)] = \frac{3(3k-2)(189k^2-345k+109)}{525k-700}.$$

Use this to eliminate $E[2Q_2(k-1)+Q_3(k-1)]$ in (51), obtaining

$$E[X_k^2] = \frac{106-96k+396k^2-756k^3}{175-525k}.$$

Now the result follows from (43) and (39). □

Corollary 5.7 *The sequence* $\left\{\frac{X_k - E[X_k]}{\sqrt{Var[X_k]}}\right\}_{k=1}^{\infty}$ *converges in distribution to the standard normal variable.*

Proof. This is immediate from Theorem 5.2, Theorem 5.1 , and Proposition 5.6. □

Acknowledgment

This work was completed during the author's sabbatical, which was made possible by a Carleton College Hewlett Mellon and Elledge Fellowship; the author thanks Carleton College for this support. The author also thanks the University of Pennsylvania Mathematics Department, and especially Jim Haglund, for their support and hospitality.

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