The Generalized Terwilliger Algebra and its Finite-dimensional Modules when $d = 2$

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Abstract

In [39] Terwilliger considered the $\mathbb{C}$-algebra generated by a given Bose Mesner algebra M and the associated dual Bose Mesner algebra M^*. This algebra is now known as the Terwilliger algebra and is usually denoted by T. Terwilliger showed that each vanishing intersection number and Krein parameter of M gives rise to a relation on certain generators of T. These relations are often called the triple product relations. They determine much of the structure of T, though not all of it in general. To illuminate the role these relations play, the current author introduced in [29] a generalization $\mathcal{T}$ of T. To go from T to $\mathcal{T}$, we replace M and M^* with a pair of dual character algebras C and C^*. The dimensions of C and C^* are equal; let $d+1$ denote this common dimension. Intuitively, $\mathcal{T}$ is the associative $\mathbb{C}$-algebra with identity generated by C and C^* subject to the analogues of Terwilliger's triple product relations. $\mathcal{T}$ is infinite-dimensional and noncommutative in general. In this paper we study $\mathcal{T}$ and its finite-dimensional modules when $d = 2$ and $\mathcal{T}$ has no "extra" vanishing intersection numbers or dual intersection numbers. In this case we show $\mathcal{T}$ is $\mathbb{C}$-algebra isomorphic to $M_3(\mathbb{C}) \oplus \mathcal{A}$, where $M_3(\mathbb{C})$ denotes the $\mathbb{C}$-algebra consisting of all 3 by 3 matrices with entries in $\mathbb{C}$ and $\mathcal{A}$ denotes the associative $\mathbb{C}$-algebra with identity generated by the symbols e and f subject to the relations $e^2 = e$ and $f^2 = f$. We find a basis for $\mathcal{A}$ and we determine the center of $\mathcal{A}$. We classify the finite-dimensional indecomposable $\mathcal{A}$-modules up to isomorphism. There are four such $\mathcal{A}$-modules in every odd dimension, and in every even dimension these modules are parameterized by a single complex number. We also classify the finite-dimensional irreducible $\mathcal{A}$-modules up to isomorphism. Using our results concerning $\mathcal{A}$, we find a basis for $\mathcal{T}$, we describe the center of $\mathcal{T}$, and we classify both the finite-dimensional indecomposable and the finite-dimensional irreducible $\mathcal{T}$-modules up to isomorphism.

1 Introduction

There is an object in algebraic combinatorics known as a Bose Mesner algebra. There are several equivalent definitions [9, 17, 33], but one that is particularly compact is the following [17, 33]. Let n denote a positive integer, let $M_n(\mathbb{C})$ denote the $\mathbb{C}$-algebra of all n by n matrices with complex entries, and let $J \in M_n(\mathbb{C})$ denote the matrix whose entries are all 1. By a Bose Mesner algebra of order n we mean a commutative subalgebra M of $M_n(\mathbb{C})$ which contains J and which is closed under transposition and entrywise multiplication. The vector space M together with entrywise multiplication is a commutative $\mathbb{C}$-algebra with identity J; we refer to this algebra as M'. To avoid dealing directly with the entrywise product, it is convenient to consider a certain subalgebra M^* of $M_n(\mathbb{C})$ which is isomorphic to M'; this algebra is constructed as follows. For all $X \in M$, let $\rho(X)$ denote the diagonal matrix in $M_n(\mathbb{C})$ whose iith entry is equal to X_{i1}, for $1 \leq i \leq n$. For example, $\rho(J) = I$, the identity matrix in $M_n(\mathbb{C})$. Observe the map $\rho : M \to M_n(\mathbb{C})$ is linear and let M^* denote the image of M under ρ. Since M is closed under entrywise multiplication and contains J, we see M^* is closed under ordinary matrix multiplication and contains I. Therefore M^* is a subalgebra of $M_n(\mathbb{C})$, and one can show $\rho : M' \to M^*$ is an isomorphism of $\mathbb{C}$-algebras [39]. The subalgebra T of $M_n(\mathbb{C})$ generated by M and M^* is known as the subconstituent algebra or the Terwilliger algebra [39]. It has been used to study P- and Q-polynomial association schemes [18, 39], group association schemes [8, 10], strongly regular graphs [42], Doob schemes [38], and association schemes over the Galois rings of characteristic four [32]. Other work involving the Terwilliger algebra can be found in [19, 20, 21, 22, 23, 24, 25, 27, 28, 30, 40, 41].

In [29] the current author introduced a generalization $\mathcal{T}$ of T. The algebra $\mathcal{T}$ is defined by generators and relations, and is infinite-dimensional and noncommutative in general. In this paper we continue our study of $\mathcal{T}$ and its finite-dimensional modules. Before stating our results, we describe $\mathcal{T}$ in more detail. To set the stage, we say a bit more about M, M^*, and T.

The algebras M and M^* each have two bases of interest to us. To obtain one basis of M, observe M' is semisimple, since it contains no nonzero nilpotent elements [37, Theorem 3.9]. Since M' is also commutative, it has a basis $A_0, \ldots, A_d$ consisting of mutually orthogonal idempotents. These matrices have all entries equal to zero or one and their sum is J. Moreover, for $0 \leq i \leq d$ there exists a positive integer k_i such that each row and column of A_i contains exactly k_i ones; this can be shown using the fact that A_i commutes with J. By definition of M we have $I \in M$ and it follows that I is one of $A_0, \ldots, A_d$; by convention we take $A_0 = I$. We define $E_i^* = \rho(A_i)$ for $0 \leq i \leq d$ and we observe $E_0^*, \ldots, E_d^*$ is a basis of mutually orthogonal idempotents of M^*. To obtain the other basis of M, we show M is semisimple. Observe M is closed under complex conjugation, since it has a basis $A_0, \ldots, A_d$ whose entries are all real. By definition M is closed under transposition, so it is closed under the conjugate transpose. Therefore M is semisimple [26, p. 157]. Since M is also commutative, it has a basis $E_0, \ldots, E_d$ consisting of mutually orthogonal idempotents. The matrix $n^{-1}J$ is a rank one idempotent and so must be one of $E_0, \ldots, E_d$; by convention we take $E_0 = n^{-1}J$. We define $A_i^* = n\rho(E_i)$ for $0 \leq i \leq d$. Observe $A_0^*, \ldots, A_d^*$ is a basis for M^* and $A_0^* = I$.

The inspiration for $\mathcal{T}$ is a result of Terwilliger concerning certain triple products in T; to describe this result, we recall two sets of parameters. Since $A_0, \ldots, A_d$ is a basis for M, there exist scalars p_{ij}^h such that

$$A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h \qquad (0 \leq i, j \leq d);$$

these are known as the intersection numbers of M. Similarly, there exist scalars p^{h*}_{ij} such that

$$A^*_i A^*_j = \sum_{h=0}^{d} p^{h*}_{ij} A^*_h \qquad (0 \le i, j \le d);$$

these are the known as the intersection numbers of M^* and also as the Krein parameters of M. Terwilliger showed in [39] that for $0 \le h, i, j \le d$ we have

$$E^*_h A_i E^*_j = 0 \qquad \text{iff} \qquad p^h_{ij} = 0$$

and

$$E_h A^*_i E_j = 0 \qquad \text{iff} \qquad p^{h*}_{ij} = 0.$$

We now describe the algebra $\mathcal{T}$. Let C denote an associative $\mathbb{C}$-algebra with a basis $x_0, \dots, x_d$ such that

$$x_i x_j = \sum_{h=0}^{d} p^h_{ij} x_h \qquad (0 \le i, j \le d). \tag{1}$$

Observe C is isomorphic to M; in fact, the linear map from M to C which maps A_i to x_i for $0 \le i \le d$ is an isomorphism of algebras. We write e_i to denote the image of E_i under this map and we observe $e_0, \dots, e_d$ is a basis for C consisting of mutually orthogonal idempotents. Similarly, let C^* denote an associative $\mathbb{C}$-algebra with a basis $x^*_0, \dots, x^*_d$ such that

$$x^*_i x^*_j = \sum_{h=0}^{d} p^{h*}_{ij} x^*_h \qquad (0 \le i, j \le d). \tag{2}$$

Then C^* is isomorphic to M^* and the linear map from M^* to C^* which maps A^*_i to x^*_i for $0 \le i \le d$ is an isomorphism of algebras. We write e^*_i to denote the image of E^*_i under this map and we observe $e^*_0, \dots, e^*_d$ is a basis for C^* consisting of mutually orthogonal idempotents. We define $\mathcal{T}$ to be the associative $\mathbb{C}$-algebra with identity generated by $x_0, \dots, x_d, x^*_0, \dots, x^*_d$ subject to the relations (1), (2), $x_0 = x^*_0$,

$$e^*_h x_i e^*_j = 0 \qquad \text{if} \qquad p^h_{ij} = 0 \qquad (0 \le h, i, j \le d), \tag{3}$$

and

$$e_h x^*_i e_j = 0 \qquad \text{if} \qquad p^{h*}_{ij} = 0 \qquad (0 \le h, i, j \le d). \tag{4}$$

The element $x_0 = x^*_0$ is the identity in $\mathcal{T}$. Intuitively, $\mathcal{T}$ is the associative $\mathbb{C}$-algebra with identity generated by C and C^* subject to the relations (3) and (4). We observe T is a homomorphic image of $\mathcal{T}$.

In our description above, the algebra $\mathcal{T}$ is constructed from a given Bose Mesner algebra. However, in some sense we only needed the algebras C and C^*. These algebras are examples of character algebras; see section 2 for a precise definition. In our main results we define $\mathcal{T}$ using character algebras; we do not assume an underlying Bose Mesner algebra.

We now describe our main results. To do this, we assume $d = 2$ and $\mathcal{T}$ has no "extra" vanishing intersection numbers or dual intersection numbers. (See section 3 for a precise definition.) We show $\mathcal{T}$ is $\mathbb{C}$-algebra isomorphic to $M_3(\mathbb{C}) \oplus \mathcal{A}$, where $\mathcal{A}$ is the associative $\mathbb{C}$-algebra with identity generated by symbols e and f subject to the relations $e^2 = e$ and $f^2 = f$. We find a basis for $\mathcal{A}$ and we describe the center of $\mathcal{A}$. Recall a module for a $\mathbb{C}$-algebra is said to be **indecomposable** whenever it is nonzero and is not a direct sum of two nonzero submodules. We classify the finite-dimensional indecomposable $\mathcal{A}$-modules up to isomorphism. There are four such $\mathcal{A}$-modules in every odd dimension, and in every even dimension these modules are parameterized by a single complex number. We also classify the finite-dimensional irreducible $\mathcal{A}$-modules up to isomorphism. Using our results concerning $\mathcal{A}$, we find a basis for $\mathcal{T}$, we describe the center of $\mathcal{T}$, we classify the finite-dimensional indecomposable $\mathcal{T}$-modules, and we classify the finite-dimensional irreducible $\mathcal{T}$-modules.

We remark that Tomiyama and Yamazaki [42] have studied T and its finite-dimensional modules when $d = 2$. In contrast, our results involve $\mathcal{T}$ and its finite-dimensional modules when $d = 2$.

We conclude this section by setting some notation. We write $\mathbb{C}$ to denote the field of complex numbers and $\mathbb{R}$ to denote the field of real numbers. From now on when we consider a matrix it will be convenient to index the rows and columns starting with zero. So for the rest of this paper we will regard matrices in $M_{d+1}(\mathbb{C})$ as having rows and columns indexed by $0, \ldots, d$. In this paper we consider modules for several different $\mathbb{C}$-algebras. Throughout, we consider only modules which are finite-dimensional.

2 The Generalized Terwilliger Algebra

In this section we recall the definition of the generalized Terwilliger algebra $\mathcal{T}$ and some results from [29] concerning $\mathcal{T}$ and its modules. We begin by recalling the notion of a character algebra, which generalizes both the Bose Mesner algebra and the dual Bose Mesner algebra. For more information on character algebras, see [5, 9, 13, 31, 34, 35, 36]. In [35, 36] a character algebra is the same object as the double algebra of a finite abelian classlike hypergroup.

Definition 2.1 *A **character algebra** $C = < X_0, \ldots, X_d >$ is a finite-dimensional associative $\mathbb{C}$-algebra together with a basis $X_0, \ldots, X_d$ having the following properties.*

1. *C is commutative.*
2. *X_0 is the multiplicative identity element of C.*
3. *Let p^h_{ij} $(0 \leq h, i, j \leq d)$ denote complex numbers such that*

$$X_i X_j = \sum_{h=0}^{d} p^h_{ij} X_h \qquad (0 \leq i, j \leq d). \tag{5}$$

   Then $p^h_{ij} \in \mathbb{R}$ for $0 \leq h, i, j \leq d$.

4. *There exist an involution $i \mapsto i'$ of $0, \ldots, d$ and positive real numbers k_i $(0 \leq i \leq d)$ such that*

$$p^0_{ij} = \delta_{ji'} k_i. \qquad (0 \leq i, j \leq d)$$

5. *The linear map $\tau : C \to C$ which satisfies $\tau(X_i) = X_{i'}$ for $0 \leq i \leq d$ is a $\mathbb{C}$-algebra isomorphism.*
6. *The linear map $\pi_0 : C \to \mathbb{C}$ which satisfies $\pi_0(X_i) = k_i$ for $0 \leq i \leq d$ is a $\mathbb{C}$-algebra homomorphism.*

*We refer to the scalars p^h_{ij} as the **intersection numbers** of C.*

Remark A character algebra whose intersection numbers are all nonnegative is essentially the same object as a table algebra. For more information on table algebras, see [1, 2, 3, 4, 6, 7, 11, 12, 14, 15, 16, 43, 44].

Let $C = < X_0, \ldots, X_d >$ denote a character algebra. We now recall the primitive idempotents of C and the matrix of eigenvalues of C. To do this, it is convenient to set

$$N = \sum_{i=0}^{d} k_i. \tag{6}$$

We refer to N as the **size** of C. By [9, Proposition 5.4, p. 92] there exists a basis $E_0, \ldots, E_d$ for C such that

$$E_i E_j = \delta_{ij} E_i \qquad (0 \leq i, j \leq d), \tag{7}$$

$$X_0 = \sum_{i=0}^{d} E_i, \tag{8}$$

and

$$E_0 = N^{-1} \sum_{i=0}^{d} X_i.$$

This basis is unique up to a permutation of $E_1, \ldots, E_d$. We refer to the elements $E_0, \ldots, E_d$ as the **primitive idempotents** of C. Since $X_0, \ldots, X_d$ and $E_0, \ldots, E_d$ are bases for C, there exists a matrix $P \in M_{d+1}(\mathbb{C})$ such that

$$X_i = \sum_{r=0}^{d} P_{ri} E_r \qquad (0 \leq i \leq d). \tag{9}$$

We refer to P as the **matrix of eigenvalues** for C (with respect to the ordering $E_0, \ldots, E_d$).

We now recall what it means for two character algebras to be dual. Let $C = < X_0, \ldots, X_d >$ and $C^* = < X_0^*, \ldots, X_d^* >$ denote character algebras. Let $E_0, \ldots, E_d$ denote an ordering of the primitive idempotents of C and let $E_0^*, \ldots, E_d^*$ denote an ordering of the primitive idempotents of C^*. Let P denote the matrix of eigenvalues for C and let P^* denote the matrix of eigenvalues for C^*. We say C and C^* are dual (with respect to the given orderings of their primitive idempotents) whenever

$$PP^* \in Span\{I\}.$$

When C and C^* are dual, the size N of C is equal to the size N^* of C^* and

$$PP^* = NI. \tag{10}$$

We now define the generalized Terwilliger algebra $\mathcal{T}$.

Definition 2.2 *Let $C = < X_0, \ldots, X_d >$ and $C^* = < X_0^*, \ldots, X_d^* >$ denote character algebras which are dual with respect to the orderings $E_0, \ldots, E_d$ and $E_0^*, \ldots, E_d^*$ of their primitive idempotents. Let $\mathcal{T}$ denote the associative $\mathbb{C}$-algebra with 1 which is generated by the symbols $x_0, \ldots, x_d, x_0^*, \ldots, x_d^*$ subject to the relations*

(T1) $x_0 = x_0^*$,

(T2) $x_i x_j = \sum_{h=0}^{d} p_{ij}^h x_h \qquad (0 \leq i, j \leq d)$,

(T2*) $x_i^* x_j^* = \sum_{h=0}^{d} p_{ij}^{h*} x_h^* \qquad (0 \leq i, j \leq d)$, (T3) $e^*_h x_i e^*_j = 0$ *if* $p^h_{ij} = 0$ $\qquad (0 \le h, i, j \le d)$,

(T3*) $e_h x^*_i e_j = 0$ *if* $p^{h*}_{ij} = 0$ $\qquad (0 \le h, i, j \le d)$.

The p^h_{ij} *are the intersection numbers of* C*, as defined in (5), and the* p^{h*}_{ij} *are the intersection numbers of* C^**. The* e_i *and* e^*_i *are defined by*

$$e_i = N^{-1} \sum_{j=0}^{d} P^*_{ji} x_j \qquad (0 \le i \le d) \tag{11}$$

and

$$e^*_i = N^{-1} \sum_{j=0}^{d} P_{ji} x^*_j \qquad (0 \le i \le d). \tag{12}$$

Here N *is as in (6), the matrix* P *is the matrix of eigenvalues of* C *defined in (9), and* P^* *is the matrix of eigenvalues of* C^**.*

Let $\mathcal{T}$ be as in Definition 2.2. By [29, Proposition 5.1] the common element $x_0 = x^*_0$ is the multiplicative identity in $\mathcal{T}$; we denote this element by 1. By [29, Proposition 10.2] the elements $x_0, \dots, x_d$ form a basis for a subalgebra of $\mathcal{T}$ which is $\mathbb{C}$-algebra isomorphic to C. By this and (8),

$$e_0 + \cdots + e_d = 1. \tag{13}$$

Similarly, by [29, Proposition 10.3] the elements $x^*_0, \dots, x^*_d$ form a basis for a subalgebra of $\mathcal{T}$ which is $\mathbb{C}$-algebra isomorphic to C^*. By this and (8),

$$e^*_0 + \cdots + e^*_d = 1.$$

We now recall a certain central idempotent of $\mathcal{T}$.

Definition 2.3 *Let* $\mathcal{T}$ *be as in Definition 2.2. By [29, Proposition 11.1],*

$$N \sum_{r=0}^{d} k_r^{-1} e^*_r e_0 e^*_r = N \sum_{j=0}^{d} k_j^{*-1} e_j e^*_0 e_j. \tag{14}$$

We write u_0 *to denote this element of* $\mathcal{T}$*. By [29, Proposition 11.4] this element is a central idempotent of* $\mathcal{T}$*. In other words,* $u_0 \neq 0$*,* $u_0^2 = u_0$*, and* $u_0 t = t u_0$ *for all* $t \in \mathcal{T}$*. For notational convenience we write* $u_1 = 1 - u_0$*. We observe that if* $u_1 \neq 0$ *then* u_1 *is also a central idempotent of* $\mathcal{T}$*.*

Since u_0 is a central idempotent of $\mathcal{T}$, the spaces $\mathcal{T}u_0$ and $\mathcal{T}u_1$ are two-sided ideals of $\mathcal{T}$ and

$$\mathcal{T} = \mathcal{T}u_0 + \mathcal{T}u_1 \qquad (\textit{direct sum}). \tag{15}$$

By [29, Theorem 12.3] the two-sided ideal $\mathcal{T}u_0$ is $\mathbb{C}$-algebra isomorphic to $M_{d+1}(\mathbb{C})$. As a result, there exists an irreducible $\mathcal{T}$-module V on which u_1 vanishes and u_0 acts as the identity. Moreover, V is unique up to isomorphism of $\mathcal{T}$-modules. We refer to V as the **primary module**. For more information on $\mathcal{T}u_0$ and the primary module, see [29].

In the next section we begin an investigation of $\mathcal{T}u_1$ and its finite-dimensional modules. To keep things simple, we will often assume the following condition holds.

Definition 2.4 *Let $\mathcal{T}$ be as in Definition 2.2. We say $\mathcal{T}$ has* ***no extra vanishing intersection numbers*** *whenever*

$$p^h_{ij} \neq 0 \qquad (1 \leq h, i, j \leq d).$$

We say $\mathcal{T}$ has ***no extra vanishing dual intersection numbers*** *whenever*

$$p^{h*}_{ij} \neq 0 \qquad (1 \leq h, i, j \leq d).$$

3 The Algebra $\mathcal{A}$

Let $\mathcal{T}$ be as in Definition 2.2, suppose $d = 2$, and suppose $\mathcal{T}$ has no extra vanishing intersection numbers or dual intersection numbers. In this section we show that under these conditions $\mathcal{T}$ is $\mathbb{C}$-algebra isomorphic to $M_3(\mathbb{C}) \oplus \mathcal{A}$, where $\mathcal{A}$ is defined as follows.

Definition 3.1 *We write $\mathcal{A}$ to denote the associative $\mathbb{C}$-algebra with 1 generated by symbols e and f subject to the relations*

(Ae) $e^2 = e$,

(Af) $f^2 = f$.

We begin with some relations involving u_1.

Proposition 3.2 *Let $\mathcal{T}$ be as in Definition 2.2 and suppose $d = 2$. Then*

(i) $e_0 u_1 = 0$,

(ii) $e^*_0 u_1 = 0$,

(iii) $e_2 u_1 = u_1 - e_1 u_1$,

(iv) $e^*_2 u_1 = u_1 - e^*_1 u_1$.

Proof. (i) By [29, Corollary 11.3] we have $e_0 u_0 = e_0$. Therefore $e_0 u_1 = e_0(1 - u_0) = 0$, as desired.
(ii) This is similar to the proof of (i).
(iii) Multiply (13) by u_1, use (i) to simplify the result and solve for $e_2 u_1$.
(iv) This is similar to the proof of (iii). □

We now give a spanning set for $\mathcal{T} u_1$.

Proposition 3.3 *Let $\mathcal{T}$ be as in Definition 2.2 and suppose $d = 2$. Then $\mathcal{T} u_1$ is spanned by*

$$\begin{gathered} u_1, \quad e^*_1 u_1, \quad e_1 u_1, \quad e_1 e^*_1 u_1, \quad e^*_1 e_1 u_1, \quad e^*_1 e_1 e^*_1 u_1, \quad e_1 e^*_1 e_1 u_1, \\ e_1 e^*_1 e_1 e^*_1 u_1, \quad e^*_1 e_1 e^*_1 e_1 u_1, \quad e^*_1 e_1 e^*_1 e_1 e^*_1 u_1, \quad e_1 e^*_1 e_1 e^*_1 e_1 u_1, \ \ldots \end{gathered} \tag{16}$$

Proof. This is immediate from Proposition 3.2, since $e_0, e_1, e_2, e^*_0, e^*_1, e^*_2$ together generate $\mathcal{T}$. □

Later on we show the sequence in (16) is a basis for $\mathcal{T} u_1$, provided $d = 2$ and $\mathcal{T}$ has no extra vanishing intersection numbers or dual intersection numbers. Our current goal, however, is to show $\mathcal{A}$ and $\mathcal{T} u_1$ are $\mathbb{C}$-algebra isomorphic in this case. To do this, we first obtain a $\mathbb{C}$-algebra homomorphism $\varphi : \mathcal{A} \to \mathcal{T} u_1$. We then display the inverse of φ. Specifically, we obtain a $\mathbb{C}$-algebra homomorphism $\phi : \mathcal{T} \to \mathcal{A}$ and show the restriction of ϕ to $\mathcal{T} u_1$ is the inverse of φ.

We begin with φ.

Proposition 3.4 *Let $\mathcal{T}$ be as in Definition 2.2, suppose $d = 2$, and suppose $u_1 \neq 0$. There exists a unique $\mathbb{C}$-algebra homomorphism $\varphi : \mathcal{A} \to \mathcal{T}u_1$ such that*

$$\varphi(e) = e_1 u_1 \tag{17}$$

and

$$\varphi(f) = e_1^* u_1. \tag{18}$$

Moreover, φ is surjective.

Proof. Since u_1 is a central idempotent and e_1 and e_1^* are idempotents, the elements $e_1 u_1$ and $e_1^* u_1$ are also idempotents. Therefore there exists a unique $\mathbb{C}$-algebra homomorphism $\varphi : \mathcal{A} \to \mathcal{T}u_1$ which satisfies (17) and (18). This map is surjective by Proposition 3.3, and since u_1 is a central idempotent. □

We now obtain ϕ.

Proposition 3.5 *Let $\mathcal{T}$ be as in Definition 2.2, suppose $d = 2$, and suppose $\mathcal{T}$ has no extra vanishing intersection numbers or dual intersection numbers. Then there exists a unique $\mathbb{C}$-algebra homomorphism $\phi : \mathcal{T} \to \mathcal{A}$ such that*

(i) $\phi(x_i) = P_{1i} e + P_{2i}(1 - e) \qquad (0 \leq i \leq 2)$,

(ii) $\phi(x_i^*) = P_{1i}^* f + P_{2i}^*(1 - f) \qquad (0 \leq i \leq 2)$.

Moreover,

(iii) $\phi(e_0) = 0, \qquad \phi(e_1) = e, \qquad \phi(e_2) = 1 - e,$

(iv) $\phi(e_0^*) = 0, \qquad \phi(e_1^*) = f, \qquad \phi(e_2^*) = 1 - f,$

(v) $\phi(u_0) = 0,$

(vi) $\phi(u_1) = 1,$

and ϕ is surjective.

Proof. For notational convenience set

$$\tilde{x}_i = P_{1i} e + P_{2i}(1 - e) \qquad (0 \leq i \leq 2) \tag{19}$$

and

$$\tilde{x}_i^* = P_{1i}^* f + P_{2i}^*(1 - f) \qquad (0 \leq i \leq 2).$$

Also for notational convenience set $\tilde{e}_0 = 0$, $\tilde{e}_1 = e$, and $\tilde{e}_2 = 1 - e$; observe

$$\tilde{e}_i \tilde{e}_j = \delta_{ij} \tilde{e}_i \qquad (0 \leq i, j \leq 2). \tag{20}$$

Similarly, set $\tilde{e}_0^* = 0$, $\tilde{e}_1^* = f$, and $\tilde{e}_2^* = 1 - f$ and observe

$$\tilde{e}_i^* \tilde{e}_j^* = \delta_{ij} \tilde{e}_i^* \qquad (0 \leq i, j \leq 2).$$

Observe

$$\tilde{x}_i = \sum_{j=0}^{2} P_{ji} \tilde{e}_j \qquad (0 \leq i \leq 2),$$

so in view of (10) we have

$$\tilde{e}_i = N^{-1} \sum_{j=0}^{2} P^*_{ji} \tilde{x}_j \qquad (0 \le i \le 2).$$

Similarly, we have

$$\tilde{e}^*_i = N^{-1} \sum_{j=0}^{2} P_{ji} \tilde{x}^*_i \qquad (0 \le i \le 2).$$

To show there exists a $\mathbb{C}$-algebra homomorphism $\phi : \mathcal{T} \to \mathcal{A}$ which satisfies (i)–(iv), we show $\tilde{x}_i$, $\tilde{x}^*_i$, $\tilde{e}_i$, and $\tilde{e}^*_i$ satisfy the relations in (T1)–(T3*).

(T1) We show $\tilde{x}_0 = \tilde{x}^*_0$. Setting $i = 0$ in (9) and comparing the result with (8) we find

$$P_{r0} = 1 \qquad (0 \le r \le d). \tag{21}$$

Setting $i = 0$ in (19) and using (21) to evaluate the result we find $\tilde{x}_0 = 1$. Similarly, $\tilde{x}^*_0 = 1$, so $\tilde{x}_0 = \tilde{x}^*_0$.

(T2) We show

$$\tilde{x}_i \tilde{x}_j = \sum_{h=0}^{2} p^h_{ij} \tilde{x}_h \qquad (0 \le i, j \le 2). \tag{22}$$

Fix $i, j (0 \le i, j \le 2)$. By (19) and (20),

$$\tilde{x}_i \tilde{x}_j = P_{1i} P_{1j} e + P_{2i} P_{2j} (1 - e). \tag{23}$$

By [29, line (27)],

$$P_{ri} P_{rj} = \sum_{h=0}^{d} p^h_{ij} P_{rh} \qquad (0 \le i, j, r \le d). \tag{24}$$

Using (24) to evaluate the coefficients of e and $1 - e$ in (23) we obtain (22), as desired.

(T2*) This is similar to the proof that the relations in (T2) hold.

(T3) Fix $h, i, j (0 \le h, i, j \le 2)$ such that $p^h_{ij} = 0$. We show $\tilde{e}^*_h \tilde{x}_i \tilde{e}^*_j = 0$. Since $\mathcal{T}$ has no extra vanishing intersection numbers, at least one of h, i, and j vanishes. First suppose $h = 0$ or $j = 0$. Then $\tilde{e}^*_h \tilde{x}_i \tilde{e}^*_j = 0$ since $\tilde{e}^*_0 = 0$. Now suppose $i = 0$. Recall $p^h_{0h} = 1$, and we assume $p^h_{0j} = 0$, so we must have $h \ne j$. Now

$$\begin{aligned} \tilde{e}^*_h \tilde{x}_i \tilde{e}^*_j &= \tilde{e}^*_h \tilde{e}^*_j && (\text{since } \tilde{x}_0 = 1) \\ &= 0 && (\text{by } (20)). \end{aligned}$$

(T3*) This is similar to the proof that the relations in (T3) hold.

We have now shown $\tilde{x}_i$, $\tilde{x}^*_i$, $\tilde{e}_i$, and $\tilde{e}^*_i$ satisfy the relations in (T1)–(T3*), so there exists a $\mathbb{C}$-algebra homomorphism $\phi : \mathcal{T} \to \mathcal{A}$ which satisfies (i) and (ii). It is unique since x_i, x^*_i generate $\mathcal{T}$. In the above construction we showed ϕ satisfies (iii) and (iv). To see ϕ satisfies (v), apply ϕ to the left side of (14) and use (iii) to evaluate the result. Now (vi) is immediate, since $u_1 = 1 - u_0$. The map ϕ is surjective by (iii) and (iv), since e and f together generate $\mathcal{A}$. □

We now show the restriction of ϕ to $\mathcal{T} u_1$ is the inverse of φ.

Proposition 3.6 *Let $\mathcal{T}$ be as in Definition 2.2, suppose $d = 2$, and suppose $\mathcal{T}$ has no extra vanishing intersection numbers or dual intersection numbers. Then the map $\varphi : \mathcal{A} \to \mathcal{T}u_1$ of Proposition 3.4 is a $\mathbb{C}$-algebra isomorphism. The inverse of φ is the restriction of ϕ to $\mathcal{T}u_1$, where ϕ is from Proposition 3.5.*

Proof. It is routine using (17), (18), and Proposition 3.5 to verify $\varphi(\phi(e_i u_1)) = e_i u_1$, $\varphi(\phi(e_i^* u_1)) = e_i^* u_1$ for $0 \le i \le 2$, as well as $\phi(\varphi(e)) = e$ and $\phi(\varphi(f)) = f$. Therefore, the restriction of ϕ to $\mathcal{T}u_1$ is the inverse of φ. It follows that φ is a $\mathbb{C}$-algebra isomorphism. □

Corollary 3.7 *Let $\mathcal{T}$ be as in Definition 2.2, suppose $d = 2$, and suppose $\mathcal{T}$ has no extra vanishing intersection numbers or dual intersection numbers. Then $\mathcal{T}$ is $\mathbb{C}$-algebra isomorphic to $M_3(\mathbb{C}) \oplus \mathcal{A}$.*

Proof. Recall the $\mathbb{C}$-algebra $\mathcal{T}u_0$ is isomorphic to $M_3(\mathbb{C})$. By Proposition 3.6 the $\mathbb{C}$-algebra $\mathcal{T}u_1$ is isomorphic to $\mathcal{A}$. Now the result follows from (15). □

In view of Corollary 3.7, we now turn our attention to $\mathcal{A}$ and its modules. We resume our study of $\mathcal{T}$ and its modules in sections 14–16.

4 A Basis for $\mathcal{A}$

Let λ denote an indeterminant and let $\mathbb{C}[\lambda]$ denote the $\mathbb{C}$-algebra consisting of all polynomials in λ with coefficients in $\mathbb{C}$. Let $M_2(\mathbb{C}[\lambda])$ denote the $\mathbb{C}$-algebra of two by two matrices with entries in $\mathbb{C}[\lambda]$. In this section we describe a certain subalgebra A of $M_2(\mathbb{C}[\lambda])$ and we give a $\mathbb{C}$-algebra isomorphism from $\mathcal{A}$ to A. We conclude by characterizing those elements of $M_2(\mathbb{C}[\lambda])$ which are also in A.

Definition 4.1 *We write*

$$E = \begin{pmatrix} 1 & \lambda \\ 0 & 0 \end{pmatrix} \tag{25}$$

and

$$F = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}. \tag{26}$$

We write A to denote the subalgebra of $M_2(\mathbb{C}[\lambda])$ generated by E and F.

It is immediate from (25) and (26) that $E^2 = E$ and $F^2 = F$. These facts lead us to the following definition.

Definition 4.2 *Let $\mathcal{A}$ be as in Definition 3.1. We write ω to denote the $\mathbb{C}$-algebra homomorphism from $\mathcal{A}$ to A which satisfies*

$$\omega(e) = E \tag{27}$$

and

$$\omega(f) = F. \tag{28}$$

Shortly we will show ω is a $\mathbb{C}$-algebra isomorphism. To do this, we use the following lemma.

Lemma 4.3 *Let E and F be as in Definition 4.1. Then*

(i) $(EF)^n = \lambda^n \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad (n \geq 1),$

(ii) $(FE)^n = \lambda^{n-1} \begin{pmatrix} 0 & 0 \\ 1 & \lambda \end{pmatrix} \qquad (n \geq 1),$

(iii) $F(EF)^n = (FE)^n F = \lambda^n F \qquad (n \geq 0),$

(iv) $E(FE)^n = (EF)^n E = \lambda^n E \qquad (n \geq 0).$

Proof. (i) Observe $EF = \lambda \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

(ii) Observe $FE = \begin{pmatrix} 0 & 0 \\ 1 & \lambda \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & \lambda \end{pmatrix}^2 = \lambda \begin{pmatrix} 0 & 0 \\ 1 & \lambda \end{pmatrix}$.

(iii) Combine (i) and $F \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = F$.

(iv) Combine (ii) and $E \begin{pmatrix} 0 & 0 \\ 1 & \lambda \end{pmatrix} = \lambda E$. □

For notational convenience we make the following definition.

Definition 4.4 *We define $\epsilon_{00}, \epsilon_{01}, \epsilon_{10}, \epsilon_{11} \in M_2(\mathbb{C}[\lambda])$ as follows.*

$$\epsilon_{00} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \epsilon_{01} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\epsilon_{10} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \epsilon_{11} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Next we present a basis for $M_2(\mathbb{C}[\lambda])$ which will be useful in dealing with A.

Proposition 4.5 *The sequence*

$$\begin{aligned} &\epsilon_{01},\ \epsilon_{00},\ I,\ F,\ E,\ EF,\ FE,\ FEF,\ EFE, \\ &EFEF,\ FEFE,\ FEFEF,\ EFEFE,\ \ldots \end{aligned} \tag{29}$$

is a basis for the vector space $M_2(\mathbb{C}[\lambda])$.

Proof. The sequence

$$\begin{aligned} &\epsilon_{01},\ \epsilon_{00},\ \epsilon_{11},\ \epsilon_{10},\ \lambda\epsilon_{01},\ \lambda\epsilon_{00},\ \lambda\epsilon_{11},\ \lambda\epsilon_{10}, \\ &\lambda^2\epsilon_{01},\ \lambda^2\epsilon_{00},\ \lambda^2\epsilon_{11},\ \lambda^2\epsilon_{10},\ \ldots \end{aligned}$$

is a basis for $M_2(\mathbb{C}[\lambda])$. With respect to this basis, the matrix representing (29) is upper triangular with all diagonal entries equal to 1. Such a matrix is invertible, so (29) is a basis for $M_2(\mathbb{C}[\lambda])$. □

The basis in (29) contains a basis for A.

Corollary 4.6 *Let E, F and A be as in Definition 4.1. Then the matrices*

$$I,\ F,\ E,\ EF,\ FE,\ FEF,\ EFE, EFEF,\ FEFE,\ FEFEF,\ EFEFE,\ \ldots \tag{30}$$

form a basis for A.

Proof. Since $E^2 = E$ and $F^2 = F$, these elements span A. They are linearly independent by Proposition 4.5. □

We now show $\mathcal{A}$ and A are $\mathbb{C}$-algebra isomorphic.

Corollary 4.7 *The map $\omega : \mathcal{A} \to A$ of Definition 4.2 is a $\mathbb{C}$-algebra isomorphism. Moreover, the sequence*

$$1,\ f,\ e,\ ef,\ fe,\ fef,\ efe, efef,\ fefe,\ fefef,\ efefe,\ \ldots \tag{31}$$

is a basis for $\mathcal{A}$.

Proof. Recall $e^2 = 2$ and $f^2 = f$, so the elements in (31) span $\mathcal{A}$. Now the result is immediate from Corollary 4.6. □

We conclude this section by characterizing the elements of $M_2(\mathbb{C}[\lambda])$ which are also in A.

Corollary 4.8 *Suppose $x \in M_2(\mathbb{C}[\lambda])$ and let $p_1(\lambda), p_2(\lambda), p_3(\lambda)$, and $p_4(\lambda)$ denote elements of $\mathbb{C}[\lambda]$ such that*

$$x = \begin{pmatrix} p_1(\lambda) & p_2(\lambda) \\ p_3(\lambda) & p_4(\lambda) \end{pmatrix}.$$

Then the following are equivalent.

(i) $x \in A$.

(ii) $p_2(0) = 0$ *and* $p_1(1) - p_2(1) + p_3(1) - p_4(1) = 0$.

Proof. (i) ⇒ (ii) Use Lemma 4.3 to see (ii) holds for the elements in (30). By Corollary 4.6, line (ii) holds for all $x \in A$.

(ii) ⇒ (i) By Proposition 4.5 and Corollary 4.6 there exist $a, b \in \mathbb{C}$ and $y \in A$ such that

$$x - y = a\epsilon_{00} + b\epsilon_{01}.$$

The elements x and y both satisfy (ii), so their difference does as well. Therefore $a = b = 0$ and $x \in A$, as desired. □

5 The Center of $\mathcal{A}$

Recall an element of a $\mathbb{C}$-algebra is called **central** whenever it commutes with all elements of that algebra. The **center** of a $\mathbb{C}$-algebra is the subalgebra consisting of all central elements of that algebra. In this section we show the center of A is $\mathbb{C}$-algebra isomorphic to $\mathbb{C}[\lambda]$, from which it follows that the center of $\mathcal{A}$ is $\mathbb{C}$-algebra isomorphic to $\mathbb{C}[\lambda]$. We begin with an element of the center of A.

Proposition 5.1 *Let E, F, and A be as in Definition 4.1. Then*

$$EF + FE - E - F + I = \lambda I. \tag{32}$$

Moreover, λI is central in A.

Proof. Line (32) is immediate from (25) and (26). The last assertion is clear. □

We now describe the center of A.

Proposition 5.2 *Let A be as in Definition 4.1. Then the following hold.*

(i) *The center of A is the subalgebra of A generated by λI.*

(ii) *The center of A is $\mathbb{C}$-algebra isomorphic to $\mathbb{C}[\lambda]$.*

Proof. (i) Clearly the subalgebra of A generated by λI is in the center of A. To show the reverse inclusion, let x denote an element of the center of A. The 10-entry of x is equal to the 10-entry of $xE - Ex$ and is therefore 0. The 01-entry of x is equal to the 01-entry of $xF - Fx$ and is therefore 0. The 11-entry of x minus the 00-entry of x is equal to the 10-entry of $xF - Fx$ and is therefore 0. It follows that $x = p(\lambda)I$ for some $p(\lambda) \in \mathbb{C}[\lambda]$.
(ii) This is immediate from (i). □

In view of Corollary 4.7 and Proposition 5.1, we make the following definition.

Definition 5.3 *Let $\mathcal{A}$ be as in Definition 3.1. We write ξ to denote the element of $\mathcal{A}$ which satisfies*

$$\xi = ef + fe - e - f + 1. \tag{33}$$

We conclude this section by describing the center of $\mathcal{A}$.

Proposition 5.4 *Let $\mathcal{A}$ be as in Definition 3.1. Then the following hold.*

(i) *The center of $\mathcal{A}$ is the subalgebra of $\mathcal{A}$ generated by ξ.*

(ii) *The center of $\mathcal{A}$ is $\mathbb{C}$-algebra isomorphic to $\mathbb{C}[\lambda]$.*

Proof. Apply ω to (33) and use (27), (28), and (32) to obtain $\omega(\xi) = \lambda I$. Now (i) and (ii) follow from Corollary 4.7 and Proposition 5.2. □

6 A Presentation of $\mathcal{A}$

In Definition 3.1 we gave a presentation of $\mathcal{A}$. In this section we give a second presentation of $\mathcal{A}$ which will be useful when we consider $\mathcal{A}$-modules.

Definition 6.1 *We write $\tilde{\mathcal{A}}$ to denote the associative $\mathbb{C}$-algebra with 1 generated by b and p subject to the following relations.*

(A1) $pb = -bp$.

(A2) $b^2 + p^2 = 1$.

We first present a $\mathbb{C}$-algebra isomorphism from $\mathcal{A}$ and $\tilde{\mathcal{A}}$.

Proposition 6.2 *Let $\mathcal{A}$ be as in Definition 3.1. There exists a $\mathbb{C}$-algebra isomorphism $\sigma : \tilde{\mathcal{A}} \to \mathcal{A}$ such that*

$$\sigma(b) = e + f - 1 \tag{34}$$

and

$$\sigma(p) = e - f. \tag{35}$$

Moreover,

$$\sigma^{-1}(e) = \frac{b + p + 1}{2}$$

and

$$\sigma^{-1}(f) = \frac{b - p + 1}{2}.$$

Proof. Using (Ae) and (Af) we find

$$(e - f)(e + f - 1) = -(e + f - 1)(e - f)$$

and

$$(e + f - 1)^2 + (e - f)^2 = 1.$$

Therefore there exists a $\mathbb{C}$-algebra homomorphism $\sigma : \tilde{\mathcal{A}} \to \mathcal{A}$ which satisfies (34) and (35).

To show σ is an isomorphism we display its inverse. Using (A1) and (A2) we find

$$\frac{(b + p + 1)^2}{4} = \frac{b + p + 1}{2}$$

and

$$\frac{(b - p + 1)^2}{4} = \frac{b - p + 1}{2}.$$

Therefore there exists a $\mathbb{C}$-algebra homomorphism $\varsigma : \mathcal{A} \to \tilde{\mathcal{A}}$ such that

$$\varsigma(e) = \frac{b + p + 1}{2}$$

and

$$\varsigma(f) = \frac{b - p + 1}{2}.$$

We routinely check $\varsigma(\sigma(b)) = b$, $\varsigma(\sigma(p)) = p$, $\sigma(\varsigma(e)) = e$, and $\sigma(\varsigma(f)) = f$, so ς is the inverse of σ. Therefore σ is an $\mathbb{C}$-algebra isomorphism, as desired. □

In view of Proposition 6.2, we make the following definition.

Definition 6.3 *Let $\mathcal{A}$ be as in Definition 3.1 and let $\tilde{\mathcal{A}}$ be as in Definition 6.1. For the rest of this paper we identify $\mathcal{A}$ and $\tilde{\mathcal{A}}$ via σ, so that*

$$b = e + f - 1, \tag{36}$$

$$p = e - f, \tag{37}$$

$$e = \frac{b + p + 1}{2}, \tag{38}$$

and

$$f = \frac{b - p + 1}{2}. \tag{39}$$

We now present some useful relationships among e, f, b, and p.

Proposition 6.4 *With reference to Definition 6.3,*

(i) $be = fb$,

(ii) $bf = eb$,

(iii) $pe = (1 - f)p$,

(iv) $pf = (1 - e)p$.

Proof. (i) Use (36) to eliminate b in be and fb and use (Ae) and (Af) to evaluate the results.
(ii) This is similar to the proof of (i).
(iii) Use (37) to eliminate p in pe and $p - fp$ and use (Ae) and (Af) to evaluate the results.
(iv) This is similar to the proof of (iii). □

Next we express ξ in terms of b and in terms of p.

Proposition 6.5 *Let ξ be as in Definition 5.3. We have*

$$\xi = b^2$$

and

$$1 - \xi = p^2.$$

Proof. These are routine using (36), (37), (Ae), and (Af). □

We conclude this section by giving a basis for $\mathcal{A}$ in terms of b, p, and ξ.

Proposition 6.6 *Let $\mathcal{A}$ be as in Definition 3.1 and let ξ be as in Definition 5.3. The sequence*

$$1,\ b,\ p,\ bp,\ \xi,\ b\xi,\ p\xi,\ bp\xi,\ \xi^2,\ b\xi^2,\ p\xi^2,\ bp\xi^2,\ \ldots \tag{40}$$

is a basis for $\mathcal{A}$.

Proof. By Corollary 4.6 the sequence in (31) is a basis for $\mathcal{A}$. Therefore the sequence

$$\begin{aligned} &1,\ e+f,\ e-f,\ fe-ef,\ fe+ef,\ efe+fef,\\ &\quad efe-fef,\ fefe-efef,\ fefe+efef,\ \ldots \end{aligned}$$

is a basis for $\mathcal{A}$. With respect to this basis, the matrix representing the sequence in (40) is upper triangular with all diagonal entries equal to 1. Such a matrix is invertible, so the sequence in (40) is a basis for $\mathcal{A}$. □

7 Linear Algebra Review: Jordan Decompositions

Our main result concerning $\mathcal{A}$-modules is Theorem 12.1, which is a classification of the finite-dimensional indecomposable $\mathcal{A}$-modules. Our proof of this result uses some facts concerning linear transformations, which we summarize in this section. We begin by setting some notation.

Throughout this section, let V denote a finite-dimensional vector space over $\mathbb{C}$ and let $\theta : V \to V$ denote a linear transformation.

We now recall the notion of a generalized eigenspace of θ.

Definition 7.1 *Let c denote an eigenvalue for θ. By the* ***generalized eigenspace*** *of θ with eigenvalue c, we mean the subspace*

$$\{v \in V \mid (\theta - c1)^i v = 0 \text{ for some } i \in \mathbb{Z}^{\geq 0}\}.$$

Proposition 7.2 *Let $c_1, \ldots, c_n$ denote the distinct eigenvalues of θ and let $V_1, \ldots, V_n$ denote the associated generalized eigenspaces. Then*

$$V = \sum_{i=1}^{n} V_i \qquad \qquad (\text{direct sum}).$$

We now recall the Jordan decomposition of V with respect to θ.

Definition 7.3 *We say a subspace $W \subseteq V$ is an* ***elementary Jordan block*** *for θ whenever the following hold.*

1. *$W \neq 0$.*
2. *$\theta W \subseteq W$.*
3. *The minimal polynomial of the restriction of θ to W has the form $(x-c)^n$ for some $c \in \mathbb{C}$, where $n = \dim W$.*
4. *There does not exist a subspace W' of V satisfying 1–3 such that $W \subseteq W'$ and $W \neq W'$.*

When 1–4 hold, we observe W is contained in the generalized eigenspace of θ with eigenvalue c.

Proposition 7.4 *Let W denote an elementary Jordan block for θ and let c denote the associated eigenvalue. Then there exists a basis $v_1, \ldots, v_n$ of W such that $(\theta - cI)v_i = v_{i+1}$ for $1 \leq i < n$ and $(\theta - cI)v_n = 0$.*

Definition 7.5 *By a* ***Jordan decomposition*** *of V (with respect to θ) we mean a sequence of subspaces $V_1, \ldots, V_n$ of V such that the following hold.*

1. *V_i is an elementary Jordan block of V with respect to θ for $1 \leq i \leq n$.*
2. *$V = \sum_{i=1}^{n} V_i$ and the sum is direct.*

Proposition 7.6 *There exists a Jordan decomposition of V with respect to θ.*

Proposition 7.7 *The number of elementary Jordan blocks in a given Jordan decomposition of V with respect to θ is independent of the decomposition.*

Proposition 7.8 *Let $V_1, \ldots, V_n$ denote a sequence of subspaces of V. Then $V_1, \ldots, V_n$ is a Jordan decomposition of V with respect to θ if and only if the following hold.*

(i) *V_i satisfies conditions 1–3 of Definition 7.3 for $1 \leq i \leq n$.*

(ii) *$V = \sum_{i=1}^{n} V_i$ and the sum is direct.*

8 The Type of an Indecomposable $\mathcal{A}$-Module

Recall a module for a $\mathbb{C}$-algebra is said to be **indecomposable** whenever it is nonzero and is not a direct sum of two nonzero submodules. In this section we introduce a parameter of an indecomposable $\mathcal{A}$-module; we call this parameter the type of the module. We begin with an element of the center of $\mathcal{A}$.

Definition 8.1 *Let $b, p \in \mathcal{A}$ be as in Definition 6.3. We write*

$$\zeta = b^2 - p^2.$$

Proposition 8.2 *The element ζ is central in $\mathcal{A}$. Moreover,*

$$\zeta = 2b^2 - 1 \tag{41}$$

and

$$\zeta = 1 - 2p^2. \tag{42}$$

Proof. The first assertion is immediate from Propositions 6.5 and 5.4(i). Lines (41) and (42) are immediate from (A2). □

The characteristic polynomial of ζ on an indecomposable $\mathcal{A}$-module has a simple form.

Proposition 8.3 *Let V denote an indecomposable $\mathcal{A}$-module and set $n = \dim V$. Then the characteristic polynomial of ζ on V has the form $(x - c)^n$ for some $c \in \mathbb{C}$.*

Proof. Let $V_1, \ldots, V_k$ denote the nonzero generalized eigenspaces of ζ. By Proposition 7.2,

$$V = \sum_{i=1}^{k} V_i \qquad\qquad (\textit{direct sum}).$$

Since ζ is central in $\mathcal{A}$, the space V_i is a submodule of V for $1 \leq i \leq k$. But V is indecomposable, so we must have $k = 1$. The result follows. □

In view of Proposition 8.3, we make the following definition.

Definition 8.4 *Let V denote an indecomposable $\mathcal{A}$-module. We refer to the scalar c of Proposition 8.3 as the* ***type*** *of V.*

Let c denote the type of an indecomposable $\mathcal{A}$-module. Certain scalars associated with c will be useful in describing V. To simplify notation, we make the following convention.

Definition 8.5 *For all $c \in \mathbb{C}$ fix square roots*

$$c_+ = \sqrt{\frac{1+c}{2}} \tag{43}$$

and

$$c_- = \sqrt{\frac{1-c}{2}}.$$

We observe $c_+ = 0$ if and only if $c = -1$. Similarly, $c_- = 0$ if and only if $c = 1$. Moreover,

$$c_+^2 + c_-^2 = 1$$

and

$$c_+^2 - c_-^2 = c.$$

Next we consider the actions of b and p on an indecomposable $\mathcal{A}$-module.

Proposition 8.6 *Let V denote an indecomposable $\mathcal{A}$-module. With reference to Definition 8.5, the following are equivalent for all $c \in \mathbb{C}$.*

(i) *The type of V is equal to c.*

(ii) *The eigenvalues of b on V are contained in $\{c_+, -c_+\}$.*

(iii) *The eigenvalues of p on V are contained in $\{c_-, -c_-\}$.*

Proof. (i) $\Rightarrow$ (ii) Let α denote an eigenvalue of b on V. Observe $2b^2 - 1 = \zeta$ by (41) so $2\alpha^2 - 1$ is an eigenvalue of ζ. Now $2\alpha^2 - 1 = c$ by Proposition 8.3 so $\alpha = c_+$ or $\alpha = -c_+$.

(ii) $\Rightarrow$ (i) Let α denote the eigenvalue of ζ on V. By (41) we see $\frac{\alpha+1}{2}$ is an eigenvalue for b^2 on V. We now see $\frac{\alpha+1}{2} = c_+^2$ and it follows from (43) that $\alpha = c$. Therefore c is the type of V by Proposition 8.3 and Definition 8.4.

(i) $\Longleftrightarrow$ (iii) This is similar to the proof of (i) $\Longleftrightarrow$ (ii). □

Corollary 8.7 *Let V denote an indecomposable $\mathcal{A}$-module of type c.*

(i) *If $c = 1$ then p is nilpotent on V. If $c \neq 1$ then p is invertible on V.*

(ii) *If $c = -1$ then b is nilpotent on V. If $c \neq -1$ then b is invertible on V.*

Proof. These are immediate from Proposition 8.6. □

In view of Proposition 8.6 we make the following definition.

Definition 8.8 *Let V denote an indecomposable $\mathcal{A}$-module of type c. When $c \neq -1$ we write V^+ (resp. V^-) to denote the generalized eigenspace of b with eigenvalue c_+ (resp. $-c_+$). We observe*

$$V = V^+ + V^- \qquad (\text{direct sum}). \tag{44}$$

When $c \neq 1$ we write V_+ (resp. V_-) to denote the generalized eigenspace of p with eigenvalue c_- (resp. $-c_-$). We observe

$$V = V_+ + V_- \qquad (\text{direct sum}).$$

We conclude this section by considering the actions of b and p on V^+, V^-, V_+, and, V_-.

Proposition 8.9 *Let V denote an indecomposable $\mathcal{A}$-module of type c. If $c \neq -1$ then*

$$pV^+ \subseteq V^-, \qquad\qquad pV^- \subseteq V^+. \tag{45}$$

If $c \neq 1$ then

$$bV_+ \subseteq V_-, \qquad\qquad bV_- \subseteq V_+.$$

Proof. To obtain the inclusion on the left in (45), first fix $v \in V^+$ and $i \in \mathbb{Z}$ such that $(b - c_+ 1)^i v = 0$. Then observe

$$\begin{aligned} (b + c_+ 1)^i pv &= (-1)^i p(b - c_+ 1)^i v && \text{(by (A1))} \\ &= 0, \end{aligned}$$

so $pv \in V^-$. We have now shown the inclusion on the left in (45) holds. The remaining assertions are obtained similarly. □

9 The Indecomposable $\mathcal{A}$-Modules of Type not ± 1

The structure of an indecomposable $\mathcal{A}$-module of type 1 or -1 is fundamentally different from that of other indecomposable $\mathcal{A}$-modules, so we will study indecomposable $\mathcal{A}$-modules on a case by case basis. In this section we classify the indecomposable $\mathcal{A}$-modules with type not equal to 1 or -1 up to isomorphism. We begin by refining Proposition 8.9 for these modules.

Proposition 9.1 *Let V denote an indecomposable $\mathcal{A}$-module with type not equal to 1 or -1. The following hold.*

(i) *The maps*

$$\begin{array}{c} V^+ \to V^- \\ v \mapsto pv \end{array} \qquad\qquad \begin{array}{c} V^- \to V^+ \\ v \mapsto pv \end{array}$$

are isomorphisms of vector spaces.

(ii) *The maps*

$$\begin{array}{c} V_+ \to V_- \\ v \mapsto bv \end{array} \qquad\qquad \begin{array}{c} V_- \to V_+ \\ v \mapsto bv \end{array}$$

are isomorphisms of vector spaces.

Proof. This is immediate from Proposition 8.9 and Corollary 8.7. □

Proposition 9.1 leads to the following result concerning the dimension of V.

Corollary 9.2 *Let V denote an indecomposable $\mathcal{A}$-module with type not equal to 1 or -1 and set $n = \dim V$. Then n is even and V^+, V^-, V_+, and V_- all have dimension $\frac{n}{2}$.*

Proof. The spaces V^+ and V^- have the same dimension by Proposition 9.1(i), so $\dim V^+ = \dim V^- = \frac{n}{2}$ by (44). The case involving V_+ and V_- is similar. □

Suppose V is an indecomposable $\mathcal{A}$-module with type not equal to 1 or -1. Next we consider Jordan decompositions of V^+, V^-, V_+, and V_-.

Proposition 9.3 *Let V denote an indecomposable $\mathcal{A}$-module with type not equal to* 1 *or* -1. *Then the following hold.*

(i) *V^+ and V^- are elementary Jordan blocks for b.*

(ii) *V_+ and V_- are elementary Jordan blocks for p.*

Proof. (i) Let $V_1, \ldots, V_m$ denote a Jordan decomposition of V^+ with respect to b. Combining (A1) with Proposition 9.1(i) we find $pV_1, \ldots, pV_m$ is a Jordan decomposition of V^- with respect to b. Combining this with (44) we find

$$V = \sum_{i=1}^{m} (V_i + pV_i) \qquad (direct\ sum). \tag{46}$$

Now fix i, $1 \leq i \leq m$; we show $V_i + pV_i$ is an $\mathcal{A}$-submodule of V. To do this, first observe $bV_i \subseteq V_i$. Combining this with (A1) we find $bpV_i \subseteq pV_i$, so $V_i + pV_i$ is closed under b. Recall $p^2 = 1 - b^2$, so $p^2 V_i \subseteq V_i$. Therefore $V_i + pV_i$ is closed under p. The elements b and p together generate $\mathcal{A}$, so $V_i + pV_i$ is an $\mathcal{A}$-submodule of V, as claimed. Since V is indecomposable, we must have $m = 1$ in (46). We conclude V^+ and V^- are elementary Jordan blocks for b.

(ii) This is similar to the proof of (i). □

We now classify the indecomposable $\mathcal{A}$-modules with type not equal to 1 or -1.

Proposition 9.4 *Fix $c \in \mathbb{C}$ such that $c \neq -1$ and $c \neq 1$ and fix $m \in \mathbb{Z}^{>0}$. Then there exists an $\mathcal{A}$-module V such that*

(i) *V is indecomposable,*

(ii) $\dim V = 2m$,

(iii) *the type of V is equal to c.*

Moreover, V is unique up to isomorphism of $\mathcal{A}$-modules. In addition, there exists a basis $v_1, \ldots, v_m$, $w_1, \ldots, w_m$ of V such that

$$bv_i = c_+ v_i + v_{i+1} \qquad (1 \leq i \leq m), \tag{47}$$

$$bw_i = -c_+ w_i + w_{i+1} \qquad (1 \leq i \leq m), \tag{48}$$

$$pv_i = (-1)^i \left((1 - c_+) w_i + w_{i+1} \right) \qquad (1 \leq i \leq m), \tag{49}$$

$$pw_i = (-1)^i \left((1 + c_+) v_i + v_{i+1} \right) \qquad (1 \leq i \leq m), \tag{50}$$

where c_+ and c_- are as in Definition 8.5 and $v_{m+1} = w_{m+1} = 0$.

Proof. First let V denote an $\mathcal{A}$-module which satisfies (i)–(iii). We show V has a basis $v_1, \ldots, v_m$, $w_1, \ldots, w_m$ which satisfies (47)–(50).

Recall V^+ is an elementary Jordan block for b and observe $\dim V^+ = m$ by Corollary 9.2. By Proposition 7.4 there exists a basis $v_1, \ldots, v_m$ of V^+ which satisfies (47). We now construct a basis $w_1, \ldots, w_m$ of V^- which satisfies (49). We do this recursively, starting with w_m. By the left part of (45) and since $c_+ \neq 1$, there exists $w_m \in V^-$ such that (49) holds for $i = m$. Similarly, given j, $1 \leq j \leq m-1$, and given $w_{j+1}, \ldots, w_m$, there exists $w_j \in V^-$ which satisfies (49) with

$i = j$. By their construction, $w_1, \ldots, w_m$ satisfy (49). Moreover, they span the image $pV^+ = V^-$, so $w_1, \ldots, w_m$ is a basis for V^-. From (44) we find $v_1, \ldots, v_m$, $w_1, \ldots, w_n$ is a basis for V.

To show (48) holds we argue by induction, starting with the case $i = m$. To begin, set $i = m$ in (49) and apply b to the result. Use (A1), (47), and (49) to evaluate the left side. Solve the resulting equation for bw_m to obtain (48) with $i = m$. Now fix j, $1 \leq j \leq m-1$, and suppose (48) holds for $i = j+1$; we show it holds for $i = j$. To do this, set $i = j$ in (49) and apply b to the result. Use (A1), (47), and (49) to evaluate the left side. Use induction to evaluate the right side. Solve the resulting equation for bw_j to obtain (48) with $i = j$. We have now shown (48) holds. The proof of (50) is similar.

We have now shown every $\mathcal{A}$-module which satisfies (i)–(iii) has a basis which satisfies (47)–(50). Therefore if such an $\mathcal{A}$-module exists then it is unique up to isomorphism of $\mathcal{A}$-modules. Next we show there exists an $\mathcal{A}$-module which satisfies (i)–(iii).

Let V denote a vector space over $\mathbb{C}$ of dimension $2m$ and let $v_1, \ldots, v_m$, $w_1, \ldots, w_m$ denote a basis for V. Let $b : V \to V$ and $p : V \to V$ denote linear transformations which satisfy (47)–(50). It is routine to verify $bp = -pb$ and $b^2 + p^2 = 1$. Therefore b and p induce an $\mathcal{A}$-module structure on V. We show the $\mathcal{A}$-module V satisfies (i)–(iii).

(i) To show V is indecomposable, we show every nonzero $\mathcal{A}$-submodule of V contains v_m. To do this, we first observe by (47) and (48) that $Span\{v_1, \ldots, v_m\}$ is an elementary Jordan block for b with eigenvalue c_+ and $Span\{w_1, \ldots, w_m\}$ is an elementary Jordan block for b with eigenvalue $-c_+$. Therefore c_+ and $-c_+$ are the only eigenvalues of b on V. Moreover, v_m and w_m are (up to scalar multiplication) the unique eigenvectors for b in V with eigenvalues c_+ and $-c_+$ respectively. Now let W denote a nonzero submodule of V. Since $\mathbb{C}$ is algebraically closed and $W \neq 0$, there exists a nonzero eigenvector for b contained in W. Let v denote such a vector and observe the associated eigenvalue is either c_+ or $-c_+$. Suppose the associated eigenvalue is c_+. Then v is a nonzero scalar multiple of v_m, so $v_m \in W$. Suppose the associated eigenvalue is $-c_+$. Then v is a nonzero scalar multiple of w_m, so $w_m \in W$. Setting $i = m$ in (50) and recalling $c_+ \neq -1$, we find $v_m \in W$. In either case $v_m \in W$, as desired.

Now let W_1 and W_2 denote nonzero submodules of V. Observe $v_m \in W_1 \cap W_2$, so the sum $W_1 + W_2$ is not direct. Therefore V is indecomposable.

(ii) This is immediate from the definition of V.

(iii) This is immediate from Proposition 8.6, since c_+ and $-c_+$ are the only eigenvalues of b on V. □

In view of Proposition 9.4, we make the following definition.

Definition 9.5 *For all $c \in \mathbb{C}$ such that $c \neq -1$ and $c \neq 1$ and all $n \in \mathbb{Z}^{>0}$ such that n is even we write $\mathcal{V}_{c,n}$ to denote the unique $\mathcal{A}$-module which satisfies (i)–(iii) of Proposition 9.4 with $n = 2m$.*

Summarizing our results so far in this section, we have the following.

Proposition 9.6 *Let V denote an $\mathcal{A}$-module. For every $c \in \mathbb{C}$ such that $c \neq -1$ and $c \neq 1$, and for every $n \in \mathbb{Z}^{>0}$, the following are equivalent.*

(i) *n is even and V is $\mathcal{A}$-module isomorphic to $\mathcal{V}_{c,n}$.*

(ii) *V is indecomposable of type c and dimension n.*

For the sake of completeness we record the following, which is clear from the proof of Proposition 9.4.

Proposition 9.7 *Fix $c \in \mathbb{C}$ such that $c \neq -1$ and $c \neq 1$, fix $m \in \mathbb{Z}^{>0}$, and set $V = \mathcal{V}_{c,2m}$. Let $v_1, \ldots, v_m$, $w_1, \ldots, w_m$ denote a basis for V which satisfies (47)–(50). Then*

$$V^+ = Span\{v_1, \ldots, v_m\}$$

and

$$V^- = Span\{w_1, \ldots, w_m\}.$$

We now describe the actions of e, f, and ξ on the basis $v_1, \ldots, v_m$, $w_1, \ldots, w_m$ of Proposition 9.4.

Proposition 9.8 *Fix $c \in \mathbb{C}$ such that $c \neq -1$ and $c \neq 1$ and fix $m \in \mathbb{Z}^{>0}$. Let $v_1, \ldots, v_m$, $w_1, \ldots, w_m$ denote a basis for $\mathcal{V}_{c,2m}$ which satisfies (47)–(50). Then for $1 \leq i \leq m$ we have*

$$2ev_i = (1 + c_+)v_i + v_{i+1} + (-1)^i \left((1 - c_+)w_i + w_{i+1}\right), \tag{51}$$

$$2ew_i = (-1)^i \left((1 + c_+)v_i + v_{i+1}\right) + (1 - c_+)w_i + w_{i+1}, \tag{52}$$

$$2fv_i = (1 + c_+)v_i + v_{i+1} + (-1)^{i+1} \left((1 - c_+)w_i + w_{i+1}\right), \tag{53}$$

$$2fw_i = (-1)^{i+1} \left((1 + c_+)v_i + v_{i+1}\right) + (1 - c_+)w_i + w_{i+1}, \tag{54}$$

$$\xi v_i = c_+^2 v_i + 2c_+ v_{i+1} + v_{i+2}, \tag{55}$$

and

$$\xi w_i = c_+^2 w_i - 2c_+ w_{i+1} + w_{i+2}. \tag{56}$$

Here $v_j = w_j = 0$ for all $j > m$, the element ξ is from (33), and c_+ is from (43).

Proof. To obtain (51), apply (38) to v_i and use (47) and (49) to evaluate the result. To obtain (52), apply (38) to w_i and use (48) and (50) to evaluate the result. To obtain (53), apply (39) to v_i and use (47) and (49) to evaluate the result. To obtain (54), apply (39) to w_i and use (48) and (50) to evaluate the result. To obtain (55), apply $\xi = b^2$ to v_i and use (47) to evaluate the result. To obtain (56), apply $\xi = b^2$ to w_i and use (48) to evaluate the result. □

10 The Indecomposable $\mathcal{A}$-Modules of Type -1

In this section we classify the indecomposable $\mathcal{A}$-modules of type -1 up to isomorphism. We begin by considering a Jordan decomposition of such an $\mathcal{A}$-module with respect to b.

Proposition 10.1 *Let V denote an indecomposable $\mathcal{A}$-module of type -1. Then V is an elementary Jordan block for b.*

Proof. Let n denote the number of elementary Jordan blocks in any Jordan decomposition of V with respect to b. We show $n = 1$. To do this, we show there exists a Jordan decomposition $V_1, \ldots, V_n$ of V with respect to b such that each of $V_1, \ldots, V_n$ is an $\mathcal{A}$-submodule of V.

Let $V_1, \ldots, V_n$ denote a Jordan decomposition of V with respect to b such that the number of spaces among $V_1, \ldots, V_n$ which are $\mathcal{A}$-submodules of V is maximal. We show each of $V_1, \ldots, V_n$ is an $\mathcal{A}$-submodule of V.

Suppose by way of contradiction at least one of $V_1, \ldots, V_n$ is not an $\mathcal{A}$-submodule of V. Reordering $V_1, \ldots, V_n$ if necessary, we may assume V_1 is not an $\mathcal{A}$-submodule of V. We now construct an $\mathcal{A}$-submodule V_1' of V such that $V_1', V_2, \ldots, V_n$ is a Jordan decomposition of V with respect to b.

Observe b is nilpotent on V by Corollary 8.7. Since V_1 is an elementary Jordan block for b, the space V_1 has a basis $v, bv, \ldots, b^{m-1}v$, where $m = \dim V_1$. The vector $b^{m-1}v$ is a nonzero element of V_1, so it is not in $V_2 + \cdots + V_n$. Observe $b^{m-1}ev + b^{m-1}(1-e)v = b^{m-1}v$, so at least one of $b^{m-1}ev$ and $b^{m-1}(1-e)v$ is not in $V_2 + \cdots + V_n$. In other words, there exists $w \in \{ev, (1-e)v\}$ such that

$$b^{m-1}w \notin V_2 + \cdots + V_n. \tag{57}$$

Set

$$V_1' = Span\{w, bw, \ldots, b^{m-1}w\}. \tag{58}$$

We show V_1' is an $\mathcal{A}$-submodule of V and $V_1', V_2, \ldots, V_n$ is a Jordan decomposition of V with respect to b.

To see V_1' is an $\mathcal{A}$-submodule of V, we show V_1' is closed under b and p. To see V_1' is closed under b, we combine $b^m v = 0$ with Proposition 6.4(i),(ii) to obtain

$$b^m w = 0. \tag{59}$$

Combining this with (58) we find V_1' is closed under b. To see V_1' is closed under p, first recall w is one of ev and $(1-e)v$. Using (38) we find $pw = w - bw$ in the first case and $pw = -w - bw$ in the second. Also recall $pb = -bp$, so $pb^i w = (-1)^i b^i pw$ for $0 \leq i \leq m$. From these facts we see V_1' is closed under p. Since b and p together generate $\mathcal{A}$, the space V_1' is an $\mathcal{A}$-submodule of V.

We now show $V_1', V_2, \ldots, V_n$ is a Jordan decomposition of V with respect to b. To do this we apply Proposition 7.8. Using (57) and (59) we routinely find $w, bw, \ldots, b^{m-1}w$ are linearly independent and $V_1' \cap (V_2 + \cdots + V_n) = 0$. Apparently $\dim V_1' = m = \dim V_1$, and $V_1 + V_2 + \cdots + V_n = V$, so $V_1' + V_2 + \cdots + V_n = V$. Using these facts, we find conditions (i) and (ii) of Proposition 7.8 hold for $V_1', V_2, \ldots, V_n$. Therefore $V_1', V_2, \ldots, V_n$ is a Jordan decomposition of V with respect to b.

We have now shown there exists an $\mathcal{A}$-submodule V_1' of V such that $V_1', V_2, \ldots, V_n$ is a Jordan decomposition of V with respect to b, contradicting our selection of $V_1, \ldots, V_n$. Therefore all of the spaces $V_1, \ldots, V_n$ are $\mathcal{A}$-submodules of V. Since V is indecomposable, we must have $n = 1$. Therefore V is an elementary Jordan block for b, as desired. $\square$

Next we consider the action of p on the kernel of b.

Proposition 10.2 *Let V denote an indecomposable $\mathcal{A}$-module of type -1. The kernel of b on V is one-dimensional and is spanned by an eigenvector for p. The associated eigenvalue is 1 or -1.*

Proof. The fact that the kernel of b on V is one-dimensional is immediate from Proposition 10.1. To see it is spanned by an eigenvector for p, fix a nonzero v in the kernel of b. Use (A1) to obtain $bpv = -pbv = 0$ and conclude pv is also in the kernel of b. Since this space is one-dimensional, v must be an eigenvector for p. By Proposition 8.6 the associated eigenvalue is equal to 1 or -1. $\square$

We now consider the case of Proposition 10.2 in which the eigenvalue of p on the kernel of b is equal to 1.

Proposition 10.3 *Fix $n \in \mathbb{Z}^{>0}$. There exists an $\mathcal{A}$-module V such that*

(i) *V is indecomposable,*

(ii) *$\dim V = n$,*

(iii) *the type of* V *is* -1,

(iv) *there exists a nonzero* $v \in V$ *such that* $bv = 0$ *and* $pv = v$.

Moreover, V *is unique up to isomorphism of* $\mathcal{A}$*-modules. In addition, there exists a basis* $v_1, \ldots, v_n$ *of* V *such that*

$$bv_i = v_{i+1} \qquad (1 \leq i \leq n) \tag{60}$$

and

$$pv_i = (-1)^{n-i}(v_i + v_{i+1}) \qquad (1 \leq i \leq n), \tag{61}$$

where $v_{n+1} = 0$.

Proof. First let V denote an $\mathcal{A}$-module which satisfies (i)–(iv). We show V has a basis which satisfies (60) and (61).

To begin, observe b is nilpotent on V by (iii) and Corollary 8.7, so $b^n V = 0$. Set

$$g = \begin{cases} e & \text{if } n \text{ is even;} \\ f & \text{if } n \text{ is odd.} \end{cases}$$

We claim $b^{n-1}gV = 0$. To show this, recall by Proposition 10.1 that V is an elementary Jordan block for b. Therefore $b^{n-1}V$ is the kernel of b on V, and this kernel is one-dimensional. Combining this with (iv), we find $(1-p)b^{n-1}V = 0$. When n is even we have

$$\begin{aligned} b^{n-1}gV &= b^{n-1}eV \\ &= b^{n-1}(b+p+1)V && \text{(by (38))} \\ &= b^{n-1}(p+1)V && \text{(since } b^nV = 0\text{)} \\ &= (1-p)b^{n-1}V && \text{(by (A1))} \\ &= 0. \end{aligned}$$

The proof that $b^{n-1}gV = 0$ for n odd is similar.

We now construct a basis for V which satisfies (60) and (61). Since $b^{n-1}V \neq 0$ and $b^{n-1}gV = 0$, there exists $v \in (1-g)V$ such that $b^{n-1}v \neq 0$. Set $v_i = b^{i-1}v$ for $1 \leq i \leq n$ and observe $b^n v = 0$. Using this fact we routinely find $v_1, \ldots, v_n$ are linearly independent. Since $\dim V = n$, we see $v_1, \ldots, v_n$ is a basis for V. By construction $v_1, \ldots, v_n$ satisfy (60). We now show $v_1, \ldots, v_n$ satisfy (61). Recall $v \in (1-g)V$ and $g^2 = g$, so $gv = 0$. Combining this with (38) (if n is even) or (39) (if n is odd) we find $pv = (-1)^{n+1}(v + bv)$. Combining this with (A1), we find $v_1, \ldots, v_n$ satisfies (61).

We have shown every $\mathcal{A}$-module satisfying (i)–(iv) has a basis which satisfies (60) and (61). Therefore such an $\mathcal{A}$-module is unique if it exists. We now show there exists an $\mathcal{A}$-module which satisfies (i)–(iv).

Let V denote a vector space over $\mathbb{C}$ of dimension n and let $v_1, \ldots, v_n$ denote a basis for V. Let b and p denote linear transformations of V which satisfy (60) and (61). It is routine to verify $bp = -pb$ and $b^2 + p^2 = 1$. Therefore b and p induce an $\mathcal{A}$-module structure on V such that (60) and (61) hold. We show this $\mathcal{A}$-module V satisfies (i)–(iv).

(i) Let K denote the kernel of b on V. To show V is indecomposable, we show $K \neq 0$ and that K is contained in every nonzero submodule of V. By (60) the space K has basis v_n, so $\dim K = 1$.

Let W denote a nonzero $\mathcal{A}$-submodule of V. Of course $b \in \mathcal{A}$, so $bW \subseteq W$. By (60) the element b is nilpotent on V, so b is nilpotent on W. Therefore $K \cap W \neq 0$. Now $K \subseteq W$ since $\dim K = 1$.

(ii) This is immediate from the construction of V.

(iii) This is immediate from (i) and Corollary 8.7(ii), since b is nilpotent on V.

(iv) Set $i = n$ in (60) and (61) to obtain $bv_n = 0$ and $pv_n = v_n$. □

In view of Proposition 10.3, we make the following definition.

Definition 10.4 *For all $n \in \mathbb{Z}^{>0}$ we write $\mathcal{V}^{+}_{-1,n}$ to denote the unique $\mathcal{A}$-module which satisfies (i)–(iv) of Proposition 10.3.*

We conclude our study of $\mathcal{V}^{+}_{-1,n}$ by describing the actions of e, f, and ξ on the basis $v_1, \ldots, v_n$ of Proposition 10.3.

Proposition 10.5 *Fix $n \in \mathbb{Z}^{>0}$ and let $v_1, \ldots, v_n$ denote a basis for $\mathcal{V}^{+}_{-1,n}$ which satisfies (60) and (61). Then for $1 \leq i \leq n$,*

$$ev_i = \begin{cases} 0 & \text{if } n - i \text{ is odd;} \\ v_i + v_{i+1} & \text{if } n - i \text{ is even;} \end{cases} \tag{62}$$

$$fv_i = \begin{cases} v_i + v_{i+1} & \text{if } n - i \text{ is odd;} \\ 0 & \text{if } n - i \text{ is even;} \end{cases} \tag{63}$$

$$\xi v_i = v_{i+2}. \tag{64}$$

Here $v_j = 0$ for all $j > n$ and ξ is from (33).

Proof. To obtain (62), apply (38) to v_i and use (60) and (61) to evaluate the result. To obtain (63), apply (39) to v_i and use (60) and (61) to evaluate the result. To obtain (64), apply $\xi = b^2$ to v_i and use (60) to evaluate the result. □

We now turn our attention to the case of Proposition 10.2 in which the eigenvalue of p on the kernel of b is equal to -1. Reversing the roles of p and $-p$ in Proposition 10.3, we obtain the following result.

Proposition 10.6 *Fix $n \in \mathbb{Z}^{>0}$. There exists an $\mathcal{A}$-module V such that*

(i) *V is indecomposable,*

(ii) *$\dim V = n$,*

(iii) *the type of V is -1,*

(iv) *there exists a nonzero $v \in V$ such that $bv = 0$ and $pv = -v$.*

Moreover, V is unique up to isomorphism of $\mathcal{A}$-modules. In addition, there exists a basis $v_1, \ldots, v_n$ of V such that

$$bv_i = v_{i+1} \qquad (1 \leq i \leq n) \tag{65}$$

and

$$pv_i = (-1)^{n-i+1}(v_i + v_{i+1}) \qquad (1 \leq i \leq n), \tag{66}$$

where $v_{n+1} = 0$.

In view of Proposition 10.6, we make the following definition.

Definition 10.7 *For all $n \in \mathbb{Z}^{>0}$ we write $\mathcal{V}^-_{-1,n}$ to denote the unique $\mathcal{A}$-module which satisfies (i)–(iv) of Proposition 10.6.*

We now state the analogue of Proposition 10.5 for $\mathcal{V}^-_{-1,n}$.

Proposition 10.8 *Fix $n \in \mathbb{Z}^{>0}$ and let $v_1, \ldots, v_n$ denote a basis for $\mathcal{V}^-_{-1,n}$ which satisfies (65) and (66). Then for $1 \leq i \leq n$,*

$$ev_i = \begin{cases} v_i + v_{i+1} & \text{if } n - i \text{ is odd;} \\ 0 & \text{if } n - i \text{ is even;} \end{cases}$$

$$fv_i = \begin{cases} 0 & \text{if } n - i \text{ is odd;} \\ v_i + v_{i+1} & \text{if } n - i \text{ is even;} \end{cases}$$

$$\xi v_i = v_{i+2}.$$

Here $v_j = 0$ for all $j > n$ and ξ is from (33).

Proof. This is similar to the proof of Proposition 10.5. □

Summarizing the results of this section, we have the following.

Proposition 10.9 *Fix $n \in \mathbb{Z}^{>0}$. The $\mathcal{A}$-modules $\mathcal{V}^+_{-1,n}$ and $\mathcal{V}^-_{-1,n}$ are nonisomorphic. Moreover, every n-dimensional indecomposable $\mathcal{A}$-module of type -1 is isomorphic to one of $\mathcal{V}^+_{-1,n}$ and $\mathcal{V}^-_{-1,n}$.*

11 The Indecomposable $\mathcal{A}$-Modules of Type 1

In this section we consider indecomposable $\mathcal{A}$-modules of type 1. This case is obtained from the type -1 case by reversing the roles of b and p. We state the analogues of the main results of section 10 without further proof.

Proposition 11.1 *Let V denote an indecomposable $\mathcal{A}$-module of type 1. Then V is an elementary Jordan block for p.*

Proposition 11.2 *Let V denote an indecomposable $\mathcal{A}$-module of type 1. The kernel of p on V is one-dimensional and is spanned by an eigenvector for b. The associated eigenvalue is 1 or -1.*

Proposition 11.3 *Fix $n \in \mathbb{Z}^{>0}$. There exists an $\mathcal{A}$-module V such that*

(i) *V is indecomposable,*

(ii) $\dim V = n$,

(iii) *the type of V is 1,*

(iv) *there exists a nonzero $v \in V$ such that $pv = 0$ and $bv = v$.*

Moreover, V is unique up to isomorphism of $\mathcal{A}$-modules. In addition, there exists a basis $v_1, \dots, v_n$ of V such that

$$bv_i = (-1)^{n-i}(v_i + v_{i+1}) \qquad (1 \le i \le n) \tag{67}$$

and

$$pv_i = v_{i+1} \qquad (1 \le i \le n), \tag{68}$$

where $v^+_{n+1} = 0$.

Definition 11.4 *For all $n \in \mathbb{Z}^{>0}$ we write $\mathcal{V}^+_{1,n}$ to denote the unique $\mathcal{A}$-module which satisfies (i)–(iv) of Proposition 11.3.*

Proposition 11.5 *Fix $n \in \mathbb{Z}^{>0}$ and let $v_1, \dots, v_n$ denote a basis for $\mathcal{V}^+_{1,n}$ which satisfies (67) and (68). Then for $1 \le i \le n$,*

$$ev_i = \begin{cases} 0 & \text{if } n-i \text{ is odd;} \\ v_i + v_{i+1} & \text{if } n-i \text{ is even;} \end{cases}$$

and

$$fv_i = \begin{cases} -v_{i+1} & \text{if } n-i \text{ is odd;} \\ v_i & \text{if } n-i \text{ is even;} \end{cases}$$

$$\xi v_i = v_i - v_{i+2}.$$

Here $v_j = 0$ for all $j > n$ and ξ is from (33).

Proposition 11.6 *Fix $n \in \mathbb{Z}^{>0}$. There exists an $\mathcal{A}$-module V such that*

(i) *V is indecomposable,*

(ii) $\dim V = n$,

(iii) *the type of V is 1,*

(iv) *there exists a nonzero $v \in V$ such that $pv = 0$ and $bv = -v$.*

Moreover, V is unique up to isomorphism of $\mathcal{A}$-modules. In addition, there exists a basis $v_1, \dots, v_n$ of V such that

$$bv_i = (-1)^{n-i+1}(v_i + v_{i+1}) \qquad (1 \le i \le n) \tag{69}$$

and

$$pv_i = v_{i+1} \qquad (1 \le i \le n), \tag{70}$$

where $v_{n+1} = 0$.

Definition 11.7 *For all $n \in \mathbb{Z}^{>0}$ we write $\mathcal{V}^-_{1,n}$ to denote the unique $\mathcal{A}$-module which satisfies (i)–(iv) of Proposition 11.6.*

Proposition 11.8 *Fix $n \in \mathbb{Z}^{>0}$ and let $v_1, \dots, v_n$ denote a basis for $\mathcal{V}^-_{1,n}$ which satisfies (69) and (70). Then for $1 \le i \le n$,*

$$ev_i = \begin{cases} v_i + v_{i+1} & \text{if } n-i \text{ is odd;} \\ 0 & \text{if } n-i \text{ is even;} \end{cases}$$

$$fv_i = \begin{cases} v_i & \text{if } n-i \text{ is odd;} \\ -v_{i+1} & \text{if } n-i \text{ is even;} \end{cases}$$

$$\xi v_i = v_i - v_{i+2}.$$

Here $v_j = 0$ for all $j > n$ and ξ is from (33).

Proposition 11.9 *Fix $n \in \mathbb{Z}^{>0}$. The $\mathcal{A}$-modules $\mathcal{V}_{1,n}^{+}$ and $\mathcal{V}_{1,n}^{-}$ are nonisomorphic. Moreover, every n-dimensional indecomposable $\mathcal{A}$-module of type 1 is isomorphic to one of $\mathcal{V}_{1,n}^{+}$ and $\mathcal{V}_{1,n}^{-}$.*

12 The Classification of the Indecomposable $\mathcal{A}$-Modules

Combining the results of sections 9–11, we obtain the following classification of the finite-dimensional indecomposable $\mathcal{A}$-modules.

Theorem 12.1 *The following are finite-dimensional indecomposable $\mathcal{A}$-modules:*

$$\begin{array}{llll} \mathcal{V}_{1,n}^{+}, \quad \mathcal{V}_{1,n}^{-}, & \mathcal{V}_{-1,n}^{+}, \quad \mathcal{V}_{-1,n}^{-}; & & n \in \mathbb{Z}^{>0}; \\ \mathcal{V}_{c,2n}; & c \in \mathbb{C}\backslash\{-1,1\}, & n \in \mathbb{Z}^{>0}, & n \text{ even}. \end{array} \tag{71}$$

Moreover, every finite-dimensional indecomposable $\mathcal{A}$-module is $\mathcal{A}$-module isomorphic to exactly one of the $\mathcal{A}$-modules in (71). We note these modules are as in Definitions 9.5, 10.4, 10.7, 11.4, and 11.7.

Proof. All of the $\mathcal{A}$-modules in (71) are indecomposable by definition. To prove the last assertion of the theorem, let V denote an indecomposable $\mathcal{A}$-module, set $n = \dim V$, and let c denote the type of V. If $c \neq -1$ and $c \neq 1$ then by Proposition 9.6 the dimension n is even and V is $\mathcal{A}$-module isomorphic to $\mathcal{V}_{c,n}$. If $c = -1$ then V is isomorphic to exactly one of $\mathcal{V}_{-1,n}^{+}$ and $\mathcal{V}_{-1,n}^{-}$ by Proposition 10.9. If $c = 1$ then V is isomorphic to exactly one of $\mathcal{V}_{1,n}^{+}$ and $\mathcal{V}_{1,n}^{-}$ by Proposition 11.9. □

13 The Classification of the Irreducible $\mathcal{A}$-Modules

Recall an $\mathcal{A}$-module V is said to be **irreducible** whenever it is nonzero and has no $\mathcal{A}$-submodules other than 0 and V. Observe every irreducible $\mathcal{A}$-module is indecomposable. We now classify the finite-dimensional irreducible $\mathcal{A}$-modules up to isomorphism.

Theorem 13.1 *The following are finite-dimensional irreducible $\mathcal{A}$-modules:*

$$\mathcal{V}_{1,1}^{+}, \quad \mathcal{V}_{1,1}^{-}, \quad \mathcal{V}_{-1,1}^{+}, \quad \mathcal{V}_{-1,1}^{-}, \quad \mathcal{V}_{c,2}; \qquad c \in \mathbb{C}\backslash\{-1,1\}.$$

Moreover, every finite-dimensional irreducible $\mathcal{A}$-module is $\mathcal{A}$-module isomorphic to exactly one of the $\mathcal{A}$-modules in (13.1).

Proof. By definition the $\mathcal{A}$-modules $\mathcal{V}_{1,1}^{+}, \mathcal{V}_{1,1}^{-}, \mathcal{V}_{-1,1}^{+}$, and $\mathcal{V}_{-1,1}^{-}$ are one-dimensional and therefore irreducible. Now fix $c \in \mathbb{C}$ such that $c \neq -1$ and $c \neq 1$. We show $\mathcal{V}_{c,2}$ is irreducible. Recall $\mathcal{V}_{c,2}$ has dimension 2, so we need only show it has no one-dimensional $\mathcal{A}$-submodules. Suppose there exists a one-dimensional $\mathcal{A}$-submodule W. Then W is spanned by a vector which is a common eigenvector for b and p. Setting $m = 1$ in (47)–(50), we routinely find $\mathcal{V}_{c,2}$ has no common eigenvector for b and p. Therefore W does not exist, so $\mathcal{V}_{c,2}$ is irreducible.

To prove the last assertion of the theorem, suppose V is an irreducible $\mathcal{A}$-module and let n denote the dimension of V. Observe V is indecomposable and let c denote the type of V. We consider three cases.

First suppose $c = -1$. By Proposition 10.2, the kernel of b on V is a one-dimensional $\mathcal{A}$-submodule of V. This kernel is equal to V by irreducibility, so $n = 1$. Now V is $\mathcal{A}$-module isomorphic to one of $\mathcal{V}_{-1,1}^{+}$ and $\mathcal{V}_{-1,1}^{-}$ by Proposition 10.9.

Next suppose $c = 1$. By Proposition 11.2, the kernel of p on V is a one-dimensional $\mathcal{A}$-submodule of V. This kernel is equal to V by irreducibility, so $n = 1$. Now V is $\mathcal{A}$-module isomorphic to one of $\mathcal{V}^+_{1,1}$ and $\mathcal{V}^-_{1,1}$ by Proposition 11.9.

Finally, suppose $c \neq -1$ and $c \neq 1$. By Proposition 9.6 we find n is even and V is $\mathcal{A}$-module isomorphic to $\mathcal{V}_{c,n}$. Set $n = 2m$ and let $v_1, \ldots, v_m, w_1, \ldots, w_m$ denote a basis for V which satisfies (47)–(50). Setting $i = m$ in (47)–(50) we find $Span\{v_m, w_m\}$ is a nonzero submodule of V. Since V is irreducible, we must have $V = Span\{v_m, w_m\}$. Therefore $m = 1$ and $n = 2$. Now V is $\mathcal{A}$-module isomorphic to $\mathcal{V}_{c,2}$. □

14 A Basis for $\mathcal{T}$ when $d = 2$

Let $\mathcal{T}$ be as in Definition 2.2, suppose $d = 2$, and suppose $\mathcal{T}$ has no extra vanishing intersection numbers or dual intersection numbers. In this section we give a basis for $\mathcal{T}$. We begin with a basis for $\mathcal{T}u_1$.

Proposition 14.1 *Let $\mathcal{T}$ be as in Definition 2.2, suppose $d = 2$, and suppose $\mathcal{T}$ has no extra vanishing intersection numbers or dual intersection numbers. The following is a basis for $\mathcal{T}u_1$.*

$$\begin{array}{c} u_1, \;\; e_1^*u_1, \;\; e_1u_1, \;\; e_1e_1^*u_1, \;\; e_1^*e_1u_1, \;\; e_1^*e_1e_1^*u_1, \;\; e_1e_1^*e_1u_1, \\ e_1e_1^*e_1e_1^*u_1, \;\; e_1^*e_1e_1^*e_1u_1, \;\; e_1^*e_1e_1^*e_1e_1^*u_1, \;\; e_1e_1^*e_1e_1^*e_1u_1, \;\ldots \end{array} \tag{72}$$

Proof. Recall by Proposition 3.6 there exists a $\mathbb{C}$-algebra isomorphism $\varphi : \mathcal{A} \to \mathcal{T}u_1$ such that $\varphi(e) = e_1u_1$ and $\varphi(f) = e_1^*u_1$. A basis for $\mathcal{A}$ is given in (31). Apply φ to this basis and recall u_1 is a central idempotent to obtain (72). The result follows. □

We now give a basis for $\mathcal{T}$.

Proposition 14.2 *Let $\mathcal{T}$ be as in Definition 2.2, suppose $d = 2$, and suppose $\mathcal{T}$ has no extra vanishing intersection numbers or dual intersection numbers. Then $\mathcal{T}$ has a basis consisting of*

$$e_ie_0^*e_j \qquad\qquad (0 \leq i, j \leq 2), \tag{73}$$

together with

$$\begin{array}{c} 1, \;\; e_1^*, \;\; e_1, \;\; e_1e_1^*, \;\; e_1^*e_1, \;\; e_1^*e_1e_1^*, \;\; e_1e_1^*e_1, \\ e_1e_1^*e_1e_1^*, \;\; e_1^*e_1e_1^*e_1, \;\; e_1^*e_1e_1^*e_1e_1^*, \;\; e_1e_1^*e_1e_1^*e_1, \;\ldots \end{array} \tag{74}$$

Proof. We first show (73) and (74) together span $\mathcal{T}$. By (15) we have $\mathcal{T} = \mathcal{T}u_0 + \mathcal{T}u_1$. By [29, Proposition 12.2] the elements listed in (73) form a basis for $\mathcal{T}u_0$. We show $\mathcal{T}u_1$ is contained in the span of (73), (74). A basis for $\mathcal{T}u_1$ is given in (72), so it suffices to show each element of this basis is in the span of (73), (74). But this is immediate since $u_1 = 1 - u_0$. We have now shown (73) and (74) together span $\mathcal{T}$.

We now show the elements in (73), (74) are linearly independent. We mentioned the elements in (73) form a basis for $\mathcal{T}u_0$, so they are linearly independent. Now suppose there exists a nontrivial linear combination of the elements in (74) which is contained in the span of (73). Multiplying by u_0 and recalling $u_0u_1 = 0$ we find the corresponding linear combination of the elements in (72) is equal to zero. This contradicts Proposition 14.1. Therefore the elements in (73), (74) are linearly independent, and the result follows. □

15 The Center of $\mathcal{T}$ when $d = 2$

Let $\mathcal{T}$ be as in Definition 2.2, suppose $d = 2$, and suppose $\mathcal{T}$ has no extra vanishing intersection numbers or dual intersection numbers. In this section we describe the center of $\mathcal{T}$. We begin by describing the center of $\mathcal{T}u_1$.

Proposition 15.1 *Let $\mathcal{T}$ be as in Definition 2.2, suppose $d = 2$, and suppose $\mathcal{T}$ has no extra vanishing intersection numbers or dual intersection numbers. Then the following hold.*

(i) *The center of $\mathcal{T}u_1$ is the subalgebra of $\mathcal{T}u_1$ generated by*

$$(e_1e_1^* + e_1^*e_1 - e_1 - e_1^* + 1)u_1.$$

(ii) *The center of $\mathcal{T}u_1$ is $\mathbb{C}$-algebra isomorphic to $\mathbb{C}[\lambda]$.*

Proof. (i) By Proposition 5.4(i) the center of $\mathcal{A}$ is generated by ξ, where ξ is from (33). Applying the $\mathbb{C}$-algebra isomorphism $\varphi : \mathcal{A} \to \mathcal{T}u_1$ of Proposition 3.6 we find the center of $\mathcal{T}u_1$ is generated by $\varphi(\xi)$. Using (17) and (18) we find $\varphi(\xi) = (e_1e_1^* + e_1^*e_1 - e_1 - e_1^* + 1)u_1$.

(ii) This is immediate from Proposition 5.4(ii), since $\mathcal{T}u_1$ is $\mathbb{C}$-algebra isomorphic to $\mathcal{A}$. $\Box$

We now describe the center of $\mathcal{T}$.

Proposition 15.2 *Let $\mathcal{T}$ be as in Definition 2.2, suppose $d = 2$, and suppose $\mathcal{T}$ has no extra vanishing intersection numbers or dual intersection numbers. Then the following hold.*

(i) *The center of $\mathcal{T}$ is the subalgebra of $\mathcal{T}$ generated by u_0 and $(e_1e_1^* + e_1^*e_1 - e_1 - e_1^* + 1)u_1$.*

(ii) *The center of $\mathcal{T}$ is $\mathbb{C}$-algebra isomorphic to $\mathbb{C} \oplus \mathbb{C}[\lambda]$.*

Proof. By (15) the center of $\mathcal{T}$ is equal to $Z_0 + Z_1$, where Z_0 denotes the center of $\mathcal{T}u_0$ and Z_1 denotes the center of $\mathcal{T}u_1$. Recall $\mathcal{T}u_0$ is $\mathbb{C}$-algebra isomorphic to $M_3(\mathbb{C})$, so $Z_0 = Span\{u_0\}$. The center Z_1 is given in Proposition 15.1. The result follows. $\Box$

16 A Classification of the Indecomposable $\mathcal{T}$-Modules and Irreducible $\mathcal{T}$-Modules when $d = 2$

Let $\mathcal{T}$ be as in Definition 2.2, suppose $d = 2$, and suppose $\mathcal{T}$ has no extra vanishing intersection numbers or dual intersection numbers. In this section we classify the finite-dimensional indecomposable $\mathcal{T}$-modules up to isomorphism and we classify the finite-dimensional irreducible $\mathcal{T}$-modules up to isomorphism. We begin with some comments concerning the relationship between $\mathcal{A}$-modules and $\mathcal{T}$-modules.

In view of (15), we may identify the $\mathcal{T}u_1$-modules with the $\mathcal{T}$-modules on which $u_0 = 0$. Recall the map $\varphi : \mathcal{A} \to \mathcal{T}u_1$ of Proposition 3.6 is a $\mathbb{C}$-algebra isomorphism. Using φ we may identify the $\mathcal{T}$-modules on which $u_0 = 0$ with the $\mathcal{A}$-modules. In concrete terms, this identification is carried out as follows. Let V denote an $\mathcal{A}$-module. When V is viewed as a $\mathcal{T}$-module on which $u_0 = 0$, the action of $\mathcal{T}$ is given by

$$\begin{array}{ccc} \mathcal{T} \times V & \to & V \\ t, v & \mapsto & \phi(t)v \end{array}$$

where ϕ is from Proposition 3.5. Conversely, let V denote a $\mathcal{T}$-module on which $u_0 = 0$. When V is viewed as an $\mathcal{A}$-module the action of $\mathcal{A}$ is given by

$$\begin{array}{ccc} \mathcal{A} \times V & \rightarrow & V \\ a, v & \mapsto & \varphi(a)v \end{array}$$

We now classify the indecomposable $\mathcal{T}$-modules.

Theorem 16.1 *Let $\mathcal{T}$ be as in Definition 2.2, suppose $d = 2$, and suppose $\mathcal{T}$ has no extra vanishing intersection numbers or dual intersection numbers. The following are finite-dimensional indecomposable $\mathcal{T}$-modules:*

$$\begin{array}{lll} & \textit{the primary module}, & \\ \mathcal{V}^{+}_{1,n}, \quad \mathcal{V}^{-}_{1,n}, & \mathcal{V}^{+}_{-1,n}, \quad \mathcal{V}^{-}_{-1,n}; & n \in \mathbb{Z}^{>0}; \\ \mathcal{V}_{c,n}; & c \in \mathbb{C}\backslash\{-1,1\}, & n \in \mathbb{Z}^{>0}, \quad n \textit{ even}. \end{array} \tag{75}$$

Moreover, every finite-dimensional indecomposable $\mathcal{T}$-module is $\mathcal{T}$-module isomorphic to exactly one of the $\mathcal{T}$-modules in (75).

Proof. The primary module is irreducible by definition, and is therefore indecomposable. The remaining modules in (75) are indecomposable as $\mathcal{A}$-modules by Theorem 12.1. They are indecomposable as $\mathcal{T}$-modules in view of the discussion preceeding the theorem.

To prove the last assertion of the theorem, suppose V is an indecomposable $\mathcal{T}$-module. Since u_0 is a central idempotent, the spaces u_0V and u_1V are $\mathcal{T}$-submodules of V and

$$V = u_0V + u_1V \qquad\qquad (\textit{direct sum}).$$

Therefore, either $u_0V = 0$ or $u_1V = 0$. If $u_1V = 0$ then $V = u_0V$. By [29, Proposition 14.5] such a $\mathcal{T}$-module is a direct sum of copies of the primary module. Since V is indecomposable there is only one summand, and V is $\mathcal{T}$-module isomorphic to the primary module. If $u_0V = 0$ then we view V as an $\mathcal{A}$-module as in the discussion preceeding the theorem, and observe V is indecomposable as an $\mathcal{A}$-module. Now the result follows from Theorem 12.1. □

We now classify the irreducible $\mathcal{T}$-modules.

Theorem 16.2 *Let $\mathcal{T}$ be as in Definition 2.2, suppose $d = 2$, and suppose $\mathcal{T}$ has no extra vanishing intersection numbers or dual intersection numbers. The following are finite-dimensional irreducible $\mathcal{T}$-modules:*

$$\begin{array}{ll} & \textit{the primary module}, \\ \mathcal{V}^{+}_{1,1}, \quad \mathcal{V}^{-}_{1,1}, \quad \mathcal{V}^{+}_{-1,1}, \quad \mathcal{V}^{-}_{-1,1}, \quad \mathcal{V}_{c,2}; & c \in \mathbb{C}\backslash\{-1,1\}. \end{array} \tag{76}$$

Moreover, every finite-dimensional irreducible $\mathcal{T}$-module is $\mathcal{T}$-module isomorphic to exactly one of the $\mathcal{T}$-modules in (76).

Proof. The primary module is irreducible by definition. The remaining $\mathcal{T}$-modules in (76) are irreducible as $\mathcal{A}$-modules by Theorem 13.1. They are irreducible as $\mathcal{T}$-modules in view of the discussion preceeding Theorem 16.1.

To prove the last assertion of the theorem, suppose V is an irreducible $\mathcal{T}$-module. Since u_0 is a central idempotent, the spaces u_0V and u_1V are $\mathcal{T}$-submodules of V and

$$V = u_0V + u_1V \qquad\qquad (direct\ sum).$$

Therefore, either $u_0V = 0$ or $u_1V = 0$. If $u_1V = 0$ then $V = u_0V$. By [29, Proposition 14.5] such a $\mathcal{T}$-module is a direct sum of copies of the primary module. Since V is irreducible there is only one summand, and V is $\mathcal{T}$-module isomorphic to the primary module. If $u_0V = 0$ then we may view V as an irreducible $\mathcal{A}$-module and the result follows from Theorem 13.1. □

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References

[1] Z. Arad and H. I. Blau, An infinite family of nonabelian simple table algebras not induced by finite nonabelian simple groups, *in* Groups St Andrews 1989, vol. 159 of London Mathematical Society Lecture Note Series, Cambridge University Press, 1991, 29–37.

[2] Z. Arad and H. I. Blau, On table algebras and applications to finite group theory, *J. Algebra*, **138** (1991), 137–185.

[3] Z. Arad, H. I. Blau, J. Erez, and E. Fisman, Real table algebras and applications to finite groups of extended Camina-Frobenius type, *J. Algebra*, **168** (1994), 615–647.

[4] Z. Arad, H. I. Blau, and E. Fisman, Table algebras and applications to products of characters in finite groups, *J. Algebra*, **138** (1991), 186–194.

[5] Z. Arad and E. Fisman, Table algebras, C-algebras, and applications to finite group theory, *Comm. Algebra*, **19** (1991), 2955–3009.

[6] Z. Arad, E. Fisman, V. Miloslavsky, and M. Muzychuk, On antisymmetric homogeneous integral table algebras of degree three. in Homogeneous Integral Table Algebras of Degree Three: a Trilogy, preprint.

[7] Z. Arad, E. Fisman, and C. M. Scoppola, Table algebras of extended Gagola type and applications to finite group theory, *in* Groups '93 Galway/St Andrews, vol. 211 of London Mathematical Society Lecture Note Series, Cambridge University Press, 1995, 13–22.

[8] J. M. P. Balmaceda and M. Oura, The Terwilliger algebras of the group association schemes of S_5 and A_5, *Kyushu J. Math.*, **48** (1994), 221–231.

[9] E. Bannai and T. Ito, "Algebraic Combinatorics I: Association Schemes", The Benjamin/Cummings Publishing Company, Inc., Menlo Park, California, 1984.

[10] E. Bannai and A. Munemasa, The Terwilliger algebras of group association schemes, *Kyushu J. Math.*, **49** (1995), 93–102.

[11] H. I. Blau, Homogeneous integral table algebras of degree three with no nontrivial linear elements. in Homogeneous Integral Table Algebras of Degree Three: a Trilogy, preprint.

[12] H. I. Blau, Integral table algebras, affine diagrams, and the analysis of degree two, *J. Algebra*, **178** (1995), 872–918.

[13] H. I. Blau, Quotient structures in C-algebras, *J. Algebra*, **175** (1995), 24–64.

[14] H. I. Blau, Integral table algebras of degree two, *Algebra Colloq.*, **4** (1997), 393–408.

[15] H. I. Blau and B. Xu, Homogeneous integral table algebras of degree three with a faithful real element. in Homogeneous Integral Table Algebras of Degree Three: a Trilogy, preprint.

[16] H. I. Blau and B. Xu, On homogeneous integral table algebras, *J. Algebra*, **199** (1998), 142–168.

[17] A. E. Brouwer, A. M. Cohen, and A. Neumaier, "Distance-Regular Graphs", Springer-Verlag, Berlin, 1989.

[18] J. S. Caughman, IV, The Terwilliger algebra for bipartite P- and Q-polynomial association schemes, *Discrete Math.*, **196** (1999), 65–95.

[19] B. V. C. Collins, The Terwilliger algebra of an almost-bipartite distance-regular graph and its antipodal cover. preprint.

[20] B. V. C. Collins, The girth of a thin distance-regular graph, *Graphs Combin.*, **13** (1997), 21–34.

[21] B. Curtin, 2-homogeneous bipartite distance-regular graphs, *Discrete Math.*, **187** (1998), 39–70.

[22] B. Curtin, Bipartite distance-regular graphs, *Graphs Combin.*, **15** (1999), 143–158.

[23] B. Curtin, Distance-regular graphs which support a spin model are thin, *Discrete Math.*, **197/198** (1999), 205–216.

[24] B. Curtin, The local structure of a bipartite distance-regular graph, *European J. Combin.*, **20** (1999), 739–758.

[25] B. Curtin and K. Nomura, Distance-regular graphs related to $U_q(sl(2))$. submitted.

[26] C. Curtis and I. Reiner, "Representation Theory of Finite Groups and Associative Algebras", Interscience, New York, 1962.

[27] G. Dickie, A note on bipartite P-polynomial association schemes and dual-bipartite Q-polynomial association schemes. preprint.

[28] G. Dickie, Twice Q-polynomial distance-regular graphs are thin, *European J. Combin.*, **16** (1995), 555–560.

[29] E. S. Egge, A generalization of the Terwilliger algebra, *J. Algebra*, **233** (2000), 213–252.

[30] J. Go, The Terwilliger algebra of the hypercube Q_D. *Graphs Combin.*, submitted.

[31] G. Hoheisel, Über Charaktere, *Monatsch. Math. Phys.*, **48** (1939), 448–456.

[32] H. Ishibashi, The Terwilliger algebras of certain association schemes over the Galois rings of characteristic 4, *Graphs Combin.*, **12** (1996), 39–54.

[33] F. Jaeger, M. Matsumoto, and K. Nomura, Bose-Mesner algebras related to type II matrices and spin models, *J. Algebraic Combin.*, **8** (1998), 39–72.

[34] Y. Kawada, Über den Dualitätssatz der Charaktere nichtcommutativer Gruppen, *Proceedings of the Physico-Mathematical Society of Japan*, **24** (1942), 97–109.

[35] J. R. McMullen, An algebraic theory of hypergroups, *Bull. Austral. Math. Soc.*, **20** (1979), 35–55.

[36] J. R. McMullen and J. F. Price, Duality for finite abelian hypergroups over splitting fields, *Bull. Austral. Math. Soc.*, **20** (1979), 57–70.

[37] D. S. Passman, "A Course in Ring Theory", Brooks/Cole Publishing Company, Pacific Grove, California, 1991.

[38] K. Tanabe, The irreducible modules of the Terwilliger algebras of Doob schemes, *J. Algebraic Combin.*, **6** (1997), 173–195.

[39] P. M. Terwilliger, The subconstituent algebra of an association scheme I, *J. Algebraic Combin.*, **1** (1992), 363–388.

[40] P. M. Terwilliger, The subconstituent algebra of an association scheme II, *J. Algebraic Combin.*, **2** (1993), 73–103.

[41] P. M. Terwilliger, The subconstituent algebra of an association scheme III, *J. Algebraic Combin.*, **2** (1993), 177–210.

[42] M. Tomiyama and N. Yamazaki, The subconstituent algebra of a strongly regular graph, *Kyushu J. Math.*, **48** (1994), 323–334.

[43] B. Xu, On a class of integral table algebras, *J. Algebra*, **178** (1995), 760–781.

[44] B. Xu, Polynomial table algebras and their covering numbers, *J. Algebra*, **176** (1995), 504–527.